PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A
MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

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Phil. Trans. R. Soc. Lond. A 1955 **248**, 225-251
doi: 10.1098/rsta.1955.0015

THE MOMENTS OF THE NUMBER OF POINTS OF A LATTICE IN A BOUNDED SET

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(*Communicated by H. Davenport, F.R.S.—Received* 6 *January* 1955—
Revised 5 *April* 1955)

A certain average value of the kth power of the number of points of a lattice, with determinant 1, in a bounded n-dimensional set S is evaluated in terms of the volumes of certain sets of dimensions $n, 2n, \ldots, kn$ associated with S, provided n is sufficiently large in comparison to k. The result is used to obtain a slight improvement of the Minkowski–Hlawka theorem.

1. Let $\rho(\mathbf{X}) = \rho(x_1, \ldots, x_n)$ be the characteristic function of a bounded n-dimensional set S, which is Jordan-measurable. Then, if Λ is any lattice, the number of points of Λ in S is

$$\rho(\Lambda) = \sum_{\mathbf{X}\in\Lambda} \rho(\mathbf{X}). \tag{1}$$

The object of this paper is to give formulae for the mean value of $\{\rho(\Lambda)\}^k$ over all lattices with determinant 1. The exponent k has to be restricted to be a positive integer such that

$$\max_{m=1,\ldots,k} [m^2(k-m)+1] < n, \tag{2}$$

if the present method is to be effective. The mean value is defined in the following way. Let $\Lambda = \Lambda(\alpha_1, \ldots, \alpha_{n-1}, \omega)$ be the lattice with determinant 1 generated by the points

$$\begin{aligned} \mathbf{A}_1 &= (\omega, 0, \ldots, 0, \alpha_1\omega^{-n+1}), \\ \mathbf{A}_2 &= (0, \omega, \ldots, 0, \alpha_2\omega^{-n+1}), \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ \mathbf{A}_{n-1} &= (0, 0, \ldots, \omega, \alpha_{n-1}\omega^{-n+1}), \\ \mathbf{A}_n &= (0, 0, \ldots, 0, \omega^{-n+1}). \end{aligned}$$

Take the mean value to be the limit

$$\lim_{\omega\to+0} \int_0^1 \ldots \int_0^1 \{\rho(\Lambda(\alpha_1, \ldots, \alpha_{n-1}, \omega))\}^k \, d\alpha_1 \ldots d\alpha_{n-1}, \tag{3}$$

if this limit exists. In a later paper* I hope to prove that the mean value defined in this way is essentially the same as the mean value over the set of all lattices defined in terms of Siegel's invariant measure (Siegel 1945).

The required formulae for the mean value of the kth power of the number of lattice points in a bounded Jordan-measurable set S are obtained as the particular cases of the following result when $\rho(\mathbf{X})$ is taken to be the characteristic function of S.

* Now accepted for publication in *Acta Mathematica* (August 1955).

Theorem 1. *Let $\rho(\mathbf{X})$ be a function which is integrable in the Riemann sense over the whole of n-dimensional space. Suppose that k satisfies the condition* (2). *Then, provided the integral is interpreted as an upper* Riemann integral, the mean value* (3) *exists and has the value*

$$\{\rho(\mathbf{O})\}^k+\left\{\int\rho(\mathbf{X})\,\mathrm{d}\mathbf{X}\right\}^k+\sum_{(\nu;\mu)}\sum_{q=1}^{\infty}\sum_{C}\left(\frac{N(C)}{q^m}\right)^n\int\ldots\int\prod_{r=1}^{m}\rho(\mathbf{X}_r)\prod_{s=1}^{k-m}\rho\left(\sum_{i=1}^{m}c_{is}\mathbf{X}_i\right)\mathrm{d}\mathbf{X}_1\ldots\mathrm{d}\mathbf{X}_m, \quad (4)$$

where the outer sum is over all divisions $(\nu;\mu)=(\nu_1,\ldots,\nu_m;\mu_1,\ldots,\mu_{k-m})$ of the numbers $1,2,\ldots,k$ into two sequences $\nu_1,\ldots,\nu_m$ and $\mu_1,\ldots,\mu_{k-m}$ with $1\leqslant m\leqslant k-1$,

$$\left.\begin{array}{l}1\leqslant\nu_1<\nu_2<\ldots<\nu_m\leqslant k,\\ 1\leqslant\mu_1<\mu_2<\ldots<\mu_{k-m}\leqslant k,\\ \nu_i\neq\mu_j,\quad \text{if}\quad 1\leqslant i\leqslant m,\quad 1\leqslant j\leqslant k-m,\end{array}\right\} \quad (5)$$

where the inner sum is over all $m\times(k-m)$ matrices C with rational elements c_{is} with lowest common denominator q and with

$$c_{is}=0 \quad \text{if} \quad \mu_s<\nu_i, \quad (6)$$

and where $N(C)$ is the number of sets of integers $a_1,\ldots,a_m$ satisfying

$$0\leqslant a_r<q \quad (r=1,\ldots,m),$$

for which the numbers

$$\sum_{i=1}^{m}c_{is}a_i \quad (s=1,\ldots,k-m)$$

are all integral.

While expression (4) may seem at first sight to be much more complicated than expression (3), it should be noted that, whereas expression (3) is a mean value of the kth power of a sum taken over an n-dimensional lattice, expression (4) only involves sums taken essentially over lattices of dimensions

$$m(k-m) \quad (m=1,\ldots,k-1).$$

Further, when n is large, the most important terms in (4) are those for which all the elements of the matrix C have one of the values 0, ±1, and there are only a finite number of these terms.

It is hoped that, despite its forbidding appearance, it will be possible to apply this result to several problems in the geometry of numbers. In this paper, attention will be confined to a single problem; theorem 1 will be used to prove the following result, which is a slight improvement of the Minkowski–Hlawka theorem† when $n\geqslant6$.

Theorem 4‡. *Let S be a bounded symmetrical set in n-dimensional space, not containing the origin and having a Jordan content V satisfying*

$$V<2+\tfrac{2}{3}[1+633(\tfrac{1}{2})^n]^{-1}.$$

Then, provided $n\geqslant6$, there is a lattice with determinant 1 having no point in S.

I am grateful to Professor Davenport and to the referee for some useful suggestions.

2. Let $\rho_1(\mathbf{X}),\ldots,\rho_k(\mathbf{X})$ be k functions, which are bounded for all $\mathbf{X}$, continuous except perhaps at $\mathbf{X}=\mathbf{O}$, and which vanish outside a bounded region. In this section the integral

$$\int_0^1\ldots\int_0^1\prod_{r=1}^{k}\rho_r(\Lambda(\alpha_1,\ldots,\alpha_{n-1},\omega))\,\mathrm{d}\alpha_1\ldots\mathrm{d}\alpha_{n-1} \quad (7)$$

* To obtain the corresponding result with the upper Riemann integrals replaced by lower Riemann integrals it suffices to apply theorem 2 to the functions $-\rho(\mathbf{X}),\rho(\mathbf{X}),\ldots,\rho(\mathbf{X})$.

† See Hlawka (1944).

‡ *Note added in proof:* W. Schmidt, in a paper to appear in part 4 of *Mh. Math. Phys.* **59** (1955), has obtained a rather better improvement of the Minkowski–Hlawka theorem by use of a completely different method; another improvement, which is even better when n is large, can be obtained by a refinement of the present paper.

will be considered, for small positive values of ω, and will be transformed into a certain sum

$$\left\{\prod_{r=1}^{k} \rho_r(\mathbf{O})\right\} + J + \sum_{(\nu;\,\mu)} \sum_{q=1}^{\infty} \sum_{C} J(\nu;\, \mu;\, C), \tag{8}$$

where the conditions on the summations are those given in the statement of theorem 1.

We may choose $R > 0$ so large that

$$\rho_r(\mathbf{X}) = 0 \quad \text{for} \quad r = 1, 2, \ldots, k,$$

for all points $\mathbf{X}$ with $\quad |\mathbf{X}| = \max\{|x_1|, \ldots, |x_n|\} \geqslant R.$

We suppose that ω is so small that $\quad \omega^{n-1} R < \tfrac{1}{2}.$

Let $\alpha = \alpha(\alpha_1, \ldots, \alpha_{n-1}, \omega)$ denote the linear transformation, transforming the point $\mathbf{U} = (u_1, \ldots, u_n)$ into the point $\mathbf{X} = \alpha\mathbf{U}$ given by

$$x_1 = \omega u_1, \quad x_2 = \omega u_2, \ldots, \quad x_{n-1} = \omega u_{n-1}, \quad x_n = \omega^{-n+1}\{\alpha_1 u_1 + \ldots + \alpha_{n-1} u_{n-1} + u_n\}.$$

Then $\quad \Lambda(\alpha_1, \ldots, \alpha_{n-1}, \omega) = \alpha\Lambda,$

where Λ is the lattice of points $\mathbf{U}$ with integral co-ordinates. So

$$\rho_r(\Lambda(\alpha_1, \ldots, \alpha_{n-1}, \omega)) = \rho_r(\alpha\Lambda) = \sum_{\mathbf{U} \in \Lambda} \rho_r(\alpha\mathbf{U}),$$

for $r = 1, \ldots, k$.

Now write

$$\mathbf{U} = (u_1, \ldots, u_{n-1}, u_n) = (\mathbf{u}, u_n),$$
$$\mathbf{X} = (x_1, \ldots, x_{n-1}, x_n) = (\mathbf{x}, x_n),$$
$$\mathbf{O} = (0, \ldots, 0, 0) = (\mathbf{o}, 0).$$

Then, if $\mathbf{X} = \alpha\mathbf{U}$, we have

$$\mathbf{x} = \omega\mathbf{u}, \; x_n = \omega^{-n+1}\{\alpha_1 u_1 + \ldots + \alpha_{n-1} u_{n-1} + u_n\}.$$

Thus we can write

$$\rho_r(\alpha\mathbf{U}) = \rho_r(\omega\mathbf{u}, \omega^{-n+1}\{\alpha_1 u_1 + \ldots + \alpha_{n-1} u_{n-1} + u_n\}),$$

and $\quad \rho_r(\Lambda(\alpha_1, \ldots, \alpha_{n-1}, \omega)) = \sum_{\mathbf{u} \in L} \sum_{u_n = -\infty}^{+\infty} \rho_r(\omega\mathbf{u}, \omega^{-n+1}\{\alpha_1 u_1 + \ldots + \alpha_{n-1} u_{n-1} + u_n\}),$

for $r = 1, \ldots, k$, where L is the lattice of all points $\mathbf{u}$ with integral co-ordinates. But the term

$$\rho_r(\omega\mathbf{u}, \omega^{-n+1}\{\alpha_1 u_1 + \ldots + \alpha_{n-1} u_{n-1} + u_n\})$$

will vanish unless $\quad \omega^{-n+1} |\alpha_1 u_1 + \ldots + \alpha_{n-1} u_{n-1} + u_n| < R,$

which implies that $\quad |\alpha_1 u_1 + \ldots + \alpha_{n-1} u_{n-1} + u_n| < R\omega^{n-1} < \tfrac{1}{2},$

so that $\quad \alpha_1 u_1 + \ldots + \alpha_{n-1} u_{n-1} + u_n = \|\alpha_1 u_1 + \ldots + \alpha_{n-1} u_{n-1}\|,$

where $\|x\|$ denotes $x - \{x\}$, where $\{x\}$ is the integer nearest to x. Thus

$$\rho_r(\Lambda(\alpha_1, \ldots, \alpha_{n-1}, \omega)) = \sum_{\mathbf{u} \in L} \rho_r(\omega\mathbf{u}, \omega^{-n+1} \|\alpha_1 u_1 + \ldots + \alpha_{n-1} u_{n-1}\|), \tag{9}$$

for $r = 1, \ldots, k$.

The results (9) show that (7) can be expressed in the form

$$\int_0^1 \cdots \int_0^1 \prod_{r=1}^{k} \rho_r(\Lambda(\alpha_1, \ldots, \alpha_{n-1}, \omega))\, d\alpha_1 \ldots d\alpha_{n-1}$$
$$= \sum_{\mathbf{u}_1 \in L, \ldots, \mathbf{u}_k \in L} \int_0^1 \cdots \int_0^1 \prod_{r=1}^{k} \rho_r(\omega\mathbf{u}_r, \omega^{-n+1} \|\alpha_1 u_1^{(r)} + \ldots + \alpha_{n-1} u_{n-1}^{(r)}\|)\, d\alpha_1 \ldots d\alpha_{n-1}, \tag{10}$$

where $\quad \mathbf{u}_r = (u_1^{(r)}, \ldots, u_{n-1}^{(r)}),$

the interchange of the order of integration and summation being justified, as only a finite number of terms in the sum have the property that they are non-zero for some values of $\alpha_1, \ldots, \alpha_{n-1}$ in the range of integration. It is convenient to write

$$I(\mathbf{u}_1, \ldots, \mathbf{u}_k) = \int_0^1 \ldots \int_0^1 \prod_{r=1}^{k} \rho_r(\omega \mathbf{u}_r, \omega^{-n+1} \| \alpha_1 u_1^{(r)} + \ldots + \alpha_{n-1} u_{n-1}^{(r)} \|) \, d\alpha_1 \ldots d\alpha_{n-1}. \tag{11}$$

We now regroup the terms of the sum

$$\sum_{\mathbf{u}_1 \in L, \ldots, \mathbf{u}_k \in L} I(\mathbf{u}_1, \ldots, \mathbf{u}_k) \tag{12}$$

to give a sum of the form (8); to avoid the use of too many small symbols, we write the summation conditions in double-line brackets to the right of the summation sign. For any $\mathbf{u}_1, \ldots, \mathbf{u}_k$ which are linearly dependent, but not all $\mathbf{o}$, there will be a unique division $(\nu; \mu)$ of the integers $1, 2, \ldots, k$ into two sequences $\nu_1, \ldots, \nu_m$ and $\mu_1, \ldots, \mu_{k-m}$ with $1 \leqslant m \leqslant k-1$, satisfying the conditions (5), such that the points

$$\mathbf{u}_{\nu_1}, \ldots, \mathbf{u}_{\nu_m}$$

are linearly independent, while for each j the point $\mathbf{u}_{\mu_j}$ is linearly dependent on

$$\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{\mu_j - 1}.$$

Then we can represent the points $\mathbf{u}_{\mu_1}, \ldots, \mathbf{u}_{\mu_{k-m}}$

uniquely in the form
$$\mathbf{u}_{\mu_j} = \sum_{i=1}^{m} c_{ij} \mathbf{u}_{\nu_i} \quad (j = 1, \ldots, k-m), \tag{13}$$

where
$$c_{ij} = 0 \quad \text{if} \quad \nu_i \geqslant \mu_j. \tag{14}$$

Here the numbers c_{ij} will be uniquely defined rational numbers. Let C denote their matrix and let q denote their lowest common denominator.

Conversely, given any division $(\nu; \mu)$ of this form and any matrix C with rational elements satisfying the conditions (14), there will be points $\mathbf{u}_1, \ldots, \mathbf{u}_k$ satisfying the conditions (13), if $\mathbf{u}_{\nu_1}, \ldots, \mathbf{u}_{\nu_m}$ can be chosen to be linearly independent points such that the points

$$\sum_{i=1}^{m} c_{ij} \mathbf{u}_{\nu_i} \quad (j = 1, \ldots, k-m),$$

have integral co-ordinates. Thus

$$\Sigma[\![\mathbf{u}_1 \in L, \ldots, \mathbf{u}_k \in L]\!] \, I(\mathbf{u}_1, \ldots, \mathbf{u}_k) = \prod_{r=1}^{k} \rho_r(\mathbf{O}) + J + \sum_{(\nu;\mu)} \sum_{q=1}^{\infty} \sum_{C} J(\nu; \mu; C), \tag{15}$$

where the conditions of summation are as in the statement of theorem 1, where

$$J = \Sigma[\![\dim\{\mathbf{u}_1, \ldots, \mathbf{u}_k\} = k]\!] \, I(\mathbf{u}_1, \ldots, \mathbf{u}_k), \tag{16}$$

and
$$J(\nu; \mu; C) = \Sigma\left[\!\!\left[\begin{array}{l} \dim\{\mathbf{u}_{\nu_1}, \ldots, \mathbf{u}_{\nu_m}\} = m, \\ \sum_{i=1}^{m} c_{ij} \mathbf{u}_{\nu_i} \in L \quad (j = 1, \ldots, k-m) \end{array}\right]\!\!\right] I(\mathbf{u}_1, \ldots, \mathbf{u}_k), \tag{17}$$

and where in $I(\mathbf{u}_1, \ldots, \mathbf{u}_k)$ the points $\mathbf{u}_{\mu_1}, \ldots, \mathbf{u}_{\mu_{k-m}}$, if any, are to be regarded as defined in terms of $\mathbf{u}_{\nu_1}, \ldots, \mathbf{u}_{\nu_m}$ by the equations (13). Here we use $\dim\{\mathbf{a}_1, \ldots, \mathbf{a}_r\}$ to denote the dimension of the linear space generated by $\mathbf{a}_1, \ldots, \mathbf{a}_r$.

It is clear from (10), (11) and (15) that we have expressed (7) in the required form (8), where J and $J(\nu; \mu; C)$ are given by (16), (17) and (11).

3. The aim of this section is to show that, provided ω is sufficiently small, the $(n-1)$-dimensional integral $I(\mathbf{u}_1, \ldots, \mathbf{u}_k)$ given by (11) can be expressed in terms of the division $(\nu; \mu)$ and the matrix C associated with the points $\mathbf{u}_1, \ldots, \mathbf{u}_k$ and a certain m-dimensional integral.

In order to simplify the notation, we confine our attention to the case when

$$\nu_1 = 1, \ldots, \nu_m = m,$$

$$\mu_1 = m+1, \ldots, \mu_{k-m} = k.$$

We also write
$$\phi_r(\mathbf{u}_r, \beta_r) = \rho_r(\omega u_1^{(r)}, \ldots, \omega u_{n-1}^{(r)}, \omega^{-n+1} \|\beta_r\|),$$

and
$$\beta_r = \alpha_1 u_1^{(r)} + \ldots + \alpha_{n-1} u_{n-1}^{(r)},$$
for $r = 1, \ldots, k$. Then

$$\begin{aligned} I(\mathbf{u}_1, \ldots, \mathbf{u}_k) &= \int_0^1 \ldots \int_0^1 \prod_{r=1}^{k} \rho_r(\omega \mathbf{u}_r, \omega^{-n+1} \|\alpha_1 u_1^{(r)} + \ldots + \alpha_{n-1} u_{n-1}^{(r)}\|)\, \mathrm{d}\alpha_1 \ldots \mathrm{d}\alpha_{n-1} \\ &= \int_0^1 \ldots \int_0^1 \prod_{r=1}^{k} \phi_r(\mathbf{u}_r, \beta_r)\, \mathrm{d}\alpha_1 \ldots \mathrm{d}\alpha_{n-1}. \end{aligned} \tag{18}$$

Since C is defined so that
$$\mathbf{u}_{m+s} = \sum_{i=1}^{m} c_{is} \mathbf{u}_i \quad (s = 1, \ldots, k-m),$$

it follows that
$$\beta_{m+s} = \sum_{i=1}^{m} c_{is} \beta_i \quad (s = 1, \ldots, k-m).$$

Thus
$$I(\mathbf{u}_1, \ldots, \mathbf{u}_k) = \int_0^1 \ldots \int_0^1 \prod_{r=1}^{m} \phi_r(\mathbf{u}_r, \beta_r) \prod_{s=1}^{k-m} \phi_{m+s}\left(\mathbf{u}_{m+s}, \sum_{i=1}^{m} c_{is}\beta_i\right) \mathrm{d}\alpha_1 \ldots \mathrm{d}\alpha_{n-1}. \tag{19}$$

Now
$$\|\beta_r\| = \|\alpha_1 u_1^{(r)} + \ldots + \alpha_{n-1} u_{n-1}^{(r)}\| \quad (r = 1, \ldots, k)$$

is periodic in $\alpha_1, \ldots, \alpha_{n-1}$ with period 1. So the integrand in (18) and (19) is periodic in $\alpha_1, \ldots, \alpha_{n-1}$ with period 1. It follows that $I(\mathbf{u}_1, \ldots, \mathbf{u}_k)$ is the limit of the mean value of the integrand

$$F(\beta_1, \ldots, \beta_m) = \prod_{r=1}^{m} \phi_r(\mathbf{u}_r, \beta_r) \prod_{s=1}^{k-m} \phi_{m+s}\left(\mathbf{u}_{m+s}, \sum_{i=1}^{m} c_{is}\beta_i\right)$$

taken over any suitable region, in the space of $\alpha_1, \ldots, \alpha_{n-1}$, which becomes large in an appropriate way. Since $\mathbf{u}_1, \ldots, \mathbf{u}_m$ are linearly independent, we may suppose that the determinant of the matrix
$$u_i^{(j)} \quad (i, j = 1, \ldots, m)$$

does not vanish. Consider the transformation from the variables $\alpha_1, \ldots, \alpha_{n-1}$ to the variables

$$\theta_r = \alpha_1 u_1^{(r)} + \ldots + \alpha_{n-1} u_{n-1}^{(r)} \quad (r = 1, \ldots, m),$$
$$\theta_r = \alpha_r \quad (r = m+1, \ldots, n-1).$$

This is a non-singular transformation. So the region defined by the inequalities

$$|\theta_r| < \Theta \quad (r = 1, \ldots, n-1)$$

is a parallepiped in the space $\alpha_1, \ldots, \alpha_{n-1}$, which becomes large as Θ tends to infinity. Thus

$$I(\mathbf{u}_1, \ldots, \mathbf{u}_k) = \lim_{\Theta\to\infty} \frac{\int_{|\theta_r|<\Theta} \cdots \int F(\theta_1, \ldots, \theta_m)\, d\alpha_1 \ldots d\alpha_{n-1}}{\int_{|\theta_r|<\Theta} \cdots \int d\alpha_1 \ldots d\alpha_{n-1}}$$
$$= \lim_{\Theta\to\infty} \frac{\int_{-\Theta}^{\Theta} \cdots \int_{-\Theta}^{\Theta} F(\theta_1, \ldots, \theta_m)\, d\theta_1 \ldots d\theta_{n-1}}{\int_{-\Theta}^{\Theta} \cdots \int_{-\Theta}^{\Theta} d\theta_1 \ldots d\theta_{n-1}}$$
$$= \lim_{\Theta\to\infty} (2\Theta)^{-m} \int_{-\Theta}^{\Theta} \cdots \int_{-\Theta}^{\Theta} F(\theta_1, \ldots, \theta_m)\, d\theta_1 \ldots d\theta_m.$$

In this formula the integrand

$$F(\theta_1, \ldots, \theta_m) = \prod_{r=1}^{m} \phi_r(\mathbf{u}_r, \theta_r) \prod_{s=1}^{k-m} \phi_{m+s}\Big(\mathbf{u}_{m+s}, \sum_{i=1}^{m} c_{is}\theta_i\Big)$$
$$= \prod_{r=1}^{m} \phi_r(\mathbf{u}_r, \|\theta_r\|) \prod_{s=1}^{k-m} \phi_{m+s}\Big(\mathbf{u}_{m+s}, \Big\|\sum_{i=1}^{m} c_{is}\theta_i\Big\|\Big)$$

is clearly periodic in $\theta_1, \ldots, \theta_m$ with period q. Consequently

$$I(\mathbf{u}_1, \ldots, \mathbf{u}_k) = \frac{1}{q^m} \int_{-\frac{1}{2}}^{q-\frac{1}{2}} \cdots \int_{-\frac{1}{2}}^{q-\frac{1}{2}} F(\theta_1, \ldots, \theta_m)\, d\theta_1 \ldots d\theta_m. \tag{20}$$

Our next object is to prove that, provided ω is sufficiently small,

$$\frac{1}{q^m} \int_{-\frac{1}{2}}^{q-\frac{1}{2}} \cdots \int_{-\frac{1}{2}}^{q-\frac{1}{2}} F(\theta_1, \ldots, \theta_m)\, d\theta_1 \ldots d\theta_m$$
$$= \frac{N(C)}{q^m} \omega^{(n-1)m} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{r=1}^{m} \rho_r(\omega\mathbf{u}_r, \xi_r) \prod_{s=1}^{k-m} \rho_{m+s}\Big(\omega\mathbf{u}_{m+s}, \sum_{i=1}^{m} c_{is}\xi_i\Big)\, d\xi_1 \ldots d\xi_m, \tag{21}$$

where $N(C)$ is defined as in the statement of theorem 1.

We first note that both sides of (21) will be zero, if any co-ordinate of any one of the points

$$\omega\mathbf{u}_1, \ldots, \omega\mathbf{u}_k$$

exceeds R. So it suffices to prove (21) under the assumption that

$$|\mathbf{u}_r| = \max\{|u_1^{(r)}|, \ldots, |u_{n-1}^{(r)}|\} \leqslant R/\omega, \tag{22}$$

for $r = 1, \ldots, k$. Now the elements c_{is} of C are determined uniquely in terms of $\mathbf{u}_1, \ldots, \mathbf{u}_k$ by the equations

$$\mathbf{u}_{m+s} = \sum_{i=1}^{m} c_{is}\mathbf{u}_i \quad (s = 1, \ldots, k-m),$$

since the points $\mathbf{u}_1, \ldots, \mathbf{u}_m$ are linearly independent. Further, if

$$u_{\lambda_j}^{(i)} \quad (i, j = 1, 2, \ldots, m)$$

is a non-singular $m \times m$ minor of the matrix

$$u_j^{(i)} \quad (i = 1, \ldots, m; j = 1, \ldots, n-1), \tag{23}$$

it is clear from Cramer's rule that any element c_{is} of C can be expressed in the form

$$c_{is} = \pm \det(u_{\lambda_j}^{(\sigma_i)}) / \det(u_{\lambda_j}^{(i)}),$$

where the matrix $(u^{(\sigma i)}_{\lambda j})$ is a minor of the matrix

$$u^{(i)}_j \quad (i = 1, \ldots, k;\, j = 1, \ldots, n-1).$$

Thus c_{is} is a rational number, whose denominator is a divisor of the determinant

$$\det (u^{(i)}_{\lambda j}).$$

So the lowest common denominator q of the elements c_{is} is a divisor of the highest common factor of the determinants of the $m \times m$ minors of the matrix (23). Thus, in particular,

$$\begin{aligned} 0 < q \leqslant |\det (u^{(i)}_{\lambda j})| &\leqslant m^{\frac{1}{2}m} \max_{\substack{1\leqslant i\leqslant m\\ 1\leqslant j\leqslant n-1}} |u^{(i)}_j|^m \\ &\leqslant m^{\frac{1}{2}m} \max_{1\leqslant i\leqslant m} |\mathbf{u}_i|^m \\ &\leqslant m^{\frac{1}{2}m} (R/\omega)^m. \end{aligned} \tag{24}$$

It also follows that
$$|qc_{is}| \leqslant |\det (u^{(\sigma i)}_{\lambda j})| \leqslant m^{\frac{1}{2}m} (R/\omega)^m. \tag{25}$$

We return to the study of the integrand $F(\theta_1, \ldots, \theta_m)$. It is clear that this will be zero unless $\theta_1, \ldots, \theta_m$ are all near to integers; indeed

$$F(\theta_1, \ldots, \theta_m) = 0$$

unless
$$\omega^{-n+1} \|\theta_r\| < R \quad \text{for} \quad r = 1, \ldots, k,$$

where we now write
$$\theta_{m+s} = \sum_{i=1}^{m} c_{is} \theta_i \quad (s = 1, \ldots, k-m).$$

So, if $F(\theta_1, \ldots, \theta_m)$ is not zero there will be integers $a_1, \ldots, a_k$ such that

$$|\theta_r - a_r| < R\omega^{n-1},$$

for $r = 1, \ldots, k$. We suppose that $R\omega^{n-1} < \frac{1}{2}$, so that the integers $a_1, \ldots, a_k$ are determined uniquely by $\theta_1, \ldots, \theta_m$. Then we have

$$0 \leqslant a_r < q \quad \text{for} \quad r = 1, \ldots, m.$$

Also, using (24) and (25),

$$\begin{aligned} \left| a_{m+s} - \sum_{i=1}^{m} c_{is} a_i \right| &\leqslant \left| \theta_{m+s} - \sum_{i=1}^{m} c_{is} a_i \right| + R\omega^{n-1} \\ &= \left| \sum_{i=1}^{m} c_{is} (\theta_i - a_i) \right| + R\omega^{n-1} \\ &\leqslant (m \max |c_{is}| + 1)\, R\omega^{n-1} \\ &\leqslant \left(\frac{m \,.\, m^{\frac{1}{2}m} (R/w)^m + q}{q} \right) R\omega^{n-1} \\ &\leqslant \frac{(m+1)\, m^{\frac{1}{2}m} (R/\omega)^m}{q} R\omega^{n-1} \\ &= \frac{(m+1)\, m^{\frac{1}{2}m} R^{m+1} \omega^{n-m-1}}{q} < \frac{1}{2q}, \end{aligned}$$

provided $k < n-1$ and ω is sufficiently small. Since the rationals c_{is} have common denominator q, this inequality implies that

$$a_{m+s} = \sum_{i=1}^{m} c_{is} a_i \quad (s = 1, \ldots, k-m).$$

Conversely, if $a_1, \ldots, a_k$ are any integers satisfying

$$\left.\begin{aligned} 0 \leqslant a_r < q \quad (r = 1, \ldots, m), \\ a_{m+s} = \sum_{i=1}^{m} c_{is} a_i \quad (s = 1, \ldots, k-m), \end{aligned}\right\} \tag{26}$$

then there will be a definite contribution to the integral (20) from the set of points $\theta_1, \ldots, \theta_m$ with

$$|a_r - \theta_r| < \tfrac{1}{2} \quad (r = 1, \ldots, m). \tag{27}$$

This contribution will be

$$\begin{aligned} \frac{1}{q^m}\int_{a_1-\frac{1}{2}}^{a_1+\frac{1}{2}} \cdots \int_{a_m-\frac{1}{2}}^{a_m+\frac{1}{2}} F(\theta_1, \ldots, \theta_m)\, d\theta_1 \ldots d\theta_m \\ = \frac{1}{q^m}\int_{-\frac{1}{2}}^{\frac{1}{2}} \cdots \int_{-\frac{1}{2}}^{\frac{1}{2}} F(\theta_1 + a_1, \ldots, \theta_m + a_m)\, d\theta_1 \ldots d\theta_m \\ = \frac{1}{q^m}\int_{-\frac{1}{2}}^{\frac{1}{2}} \cdots \int_{-\frac{1}{2}}^{\frac{1}{2}} \prod_{r=1}^{m} \phi_r(\mathbf{u}_r, \theta_r + a_r) \prod_{s=1}^{k-m} \phi_{m+s}\left(\mathbf{u}_{m+s}, \sum_{i=1}^{m} c_{is}\theta_i + a_{m+s}\right) d\theta_1 \ldots d\theta_m \\ = \frac{1}{q^m}\int_{-\frac{1}{2}}^{\frac{1}{2}} \cdots \int_{-\frac{1}{2}}^{\frac{1}{2}} \prod_{r=1}^{m} \phi_r(\mathbf{u}_r, \theta_r) \prod_{s=1}^{k-m} \phi_{m+s}\left(\mathbf{u}_{m+s}, \sum_{i=1}^{m} c_{is}\theta_i\right) d\theta_1 \ldots d\theta_m. \end{aligned}$$

Write

$$\psi_r(\mathbf{u}, \theta) = \begin{cases} \phi_r(\mathbf{u}, \theta) & \text{if } |\theta| < \tfrac{1}{2}, \\ 0 & \text{if } |\theta| \geqslant \tfrac{1}{2}. \end{cases}$$

Then the contribution becomes

$$\frac{1}{q^m}\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{r=1}^{m} \psi_r(\mathbf{u}_r, \theta_r) \prod_{s=1}^{k-m} \phi_{m+s}\left(\mathbf{u}_{m+s}, \sum_{i=1}^{m} c_{is}\theta_i\right) d\theta_1 \ldots d\theta_m.$$

But the condition

$$\prod_{r=1}^{m} \psi_r(\mathbf{u}_r, \theta_r) \neq 0$$

implies that

$$|\theta_r| \leqslant R\omega^{n-1} \quad (r = 1, \ldots, m),$$

so that, using (25),

$$\left|\sum_{i=1}^{m} c_{is}\theta_i\right| < m \max |c_{is}|\, R\omega^{n-1} < \tfrac{1}{2},$$

and

$$\phi_{m+s}\left(\mathbf{u}_{m+s}, \sum_{i=1}^{m} c_{is}\theta_i\right) = \psi_{m+s}\left(\mathbf{u}_{m+s}, \sum_{i=1}^{m} c_{is}\theta_i\right)$$

for $s = 1, \ldots, k-m$. Hence the contribution is

$$\begin{aligned} \frac{1}{q^m}\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{r=1}^{m} \psi_r(\mathbf{u}_r, \theta_r) \prod_{s=1}^{k-m} \psi_{m+s}\left(\mathbf{u}_{m+s}, \sum_{i=1}^{m} c_{is}\theta_i\right) d\theta_1 \ldots d\theta_m \\ = \frac{1}{q^m}\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{r=1}^{m} \rho_r(\omega\mathbf{u}_r, \omega^{-n+1}\theta_r) \prod_{s=1}^{k-m} \rho_{m+s}\left(\omega\mathbf{u}_{m+s}, \sum_{i=1}^{m} c_{is}\omega^{-n+1}\theta_i\right) d\theta_1 \ldots d\theta_m \\ = \left(\frac{\omega^{n-1}}{q}\right)^m \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{r=1}^{m} \rho_r(\omega\mathbf{u}_r, \xi_r) \prod_{s=1}^{k-m} \rho_{m+s}\left(\omega\mathbf{u}_{m+s}, \sum_{i=1}^{m} c_{is}\xi_i\right) d\xi_1 \ldots d\xi_m. \end{aligned}$$

Thus the total contribution to the integral

$$I(\mathbf{u}_1, \ldots, \mathbf{u}_k) = \frac{1}{q^m}\int_{-\frac{1}{2}}^{q-\frac{1}{2}} \cdots \int_{-\frac{1}{2}}^{q-\frac{1}{2}} F(\theta_1, \ldots, \theta_m)\, d\theta_1 \ldots d\theta_m,$$

from the different ranges of variation for $\theta_1, \ldots, \theta_m$ of the type (27), is

$$\frac{N(C)}{q^m}\omega^{(n-1)m}\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty}\prod_{r=1}^{m}\rho_r(\omega\mathbf{u}_r,\xi_r)\prod_{s=1}^{k-m}\rho_{m+s}\left(\omega\mathbf{u}_{m+s},\sum_{i=1}^{m}c_{is}\xi_i\right)\mathrm{d}\xi_1\ldots\mathrm{d}\xi_m,$$

where $N(C)$ is the number of sets of integers $a_1, \ldots, a_k$ satisfying the conditions (26). Since the conditions (26) determine the numbers $a_{m+1}, \ldots, a_k$ uniquely in terms of the numbers $a_1, \ldots, a_m$, it is clear that $N(C)$ is the number of sets of integers $a_1, \ldots, a_m$ with

$$0 \leqslant a_r < q \quad (r = 1, \ldots, m),$$

for which the numbers
$$\sum_{i=1}^{m} c_{is}a_i \quad (s = 1, \ldots, k-m)$$

are integral. Thus this definition of $N(C)$ is in conformity with that given in the statement of theorem 1, and (21) is established.

In the general case when the division $(\nu; \mu)$ associated with the points $\mathbf{u}_1, \ldots, \mathbf{u}_k$ is not necessarily of the special type considered above the result takes the form that

$$I(\mathbf{u}_1,\ldots,\mathbf{u}_k) = \frac{N(C)}{q^m}\omega^{(n-1)m}\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty}\prod_{i=1}^{m}\rho_{\nu_i}(\omega\mathbf{u}_{\nu_i},\xi_{\nu_i})\prod_{j=1}^{k-m}\rho_{\mu_j}\left(\omega\mathbf{u}_{\mu_j},\sum_{i=1}^{m}c_{ij}\xi_{\nu_i}\right)\mathrm{d}\xi_{\nu_1}\ldots\mathrm{d}\xi_{\nu_m}, \tag{28}$$

provided ω is sufficiently small.

We note that we have proved that $I(\mathbf{u}_1, \ldots, \mathbf{u}_k)$ will be zero for the special type of division unless the conditions (22), (24) and (25) are satisfied. In the general case it is clear that $I(\mathbf{u}_1, \ldots, \mathbf{u}_k)$ will be zero unless

$$|\mathbf{u}_{\nu_i}| \leqslant R/\omega \quad (i = 1, \ldots, m), \tag{29}$$

$$0 < q \leqslant m^{\frac{1}{2}m}\max_{1\leqslant i\leqslant m}|\mathbf{u}_{\nu_i}|^m \leqslant m^{\frac{1}{2}m}(R/\omega)^m, \tag{30}$$

and
$$|qc_{is}| \leqslant m^{\frac{1}{2}m}(R/\omega)^m \quad (i = 1, \ldots, m;\ s = 1, \ldots, k-m). \tag{31}$$

In the particular case, when the points $\mathbf{u}_1, \ldots, \mathbf{u}_k$ are linearly independent, it is easy to see that the result (20) holds with $m = k$ and $q = 1$. This leads without difficulty to the result that

$$\begin{aligned} I(\mathbf{u}_1,\ldots,\mathbf{u}_k) &= \int_{-\frac{1}{2}}^{\frac{1}{2}}\ldots\int_{-\frac{1}{2}}^{\frac{1}{2}}F(\theta_1,\ldots,\theta_k)\,\mathrm{d}\theta_1\ldots\mathrm{d}\theta_k \\ &= \omega^{(n-1)k}\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty}\prod_{r=1}^{k}\rho_r(\omega\mathbf{u}_r,\xi_r)\,\mathrm{d}\xi_1\ldots\mathrm{d}\xi_k, \end{aligned} \tag{32}$$

provided ω is sufficiently small. Here it is clear that $I(\mathbf{u}_1, \ldots, \mathbf{u}_k)$ will be zero unless

$$|\mathbf{u}_r| \leqslant R/\omega \quad (r = 1, \ldots, k). \tag{33}$$

4. This section will prove two elementary lemmas about lattice points. These lemmas will be of use later.

LEMMA 1. *The number of sets of points* $\mathbf{u}_1, \ldots, \mathbf{u}_m$ *satisfying the conditions that*

$$|\mathbf{u}_1| \leqslant R/\omega, \ldots, |\mathbf{u}_m| \leqslant R/\omega,$$

and that $\mathbf{u}_1, \ldots, \mathbf{u}_m$ *are linearly dependent, is at most*

$$m\left[1+\frac{2R}{\omega}\right]^{(m-1)n}. \tag{34}$$

Proof. If $\mathbf{u}_1, \ldots, \mathbf{u}_m$ are linearly dependent, then one of these points is linearly dependent on the other points. So the total number of sets of points satisfying the conditions is at most mN, where N is the number of such sets in which $\mathbf{u}_m$ is linearly dependent on $\mathbf{u}_1, \ldots, \mathbf{u}_{m-1}$.

Suppose now that $\mathbf{u}_1, \ldots, \mathbf{u}_{m-1}$ are fixed and that $\mathbf{u}_m$ is linearly dependent on $\mathbf{u}_1, \ldots, \mathbf{u}_{m-1}$. We may choose $\nu_1, \ldots, \nu_{m-1}$ so that the matrix

$$u^{(j)}_{\nu_i} \quad (i, j = 1, \ldots, m-1)$$

has the same rank as the matrix

$$u^{(j)}_i \quad (i = 1, \ldots, n-1;\ j = 1, \ldots, m-1).$$

Then once the co-ordinates $\qquad u^{(m)}_{\nu_i} \quad (i = 1, \ldots, m-1)$

have been fixed, the point $\mathbf{u}_m$ will be determined by the condition that it is linearly dependent on $\mathbf{u}_1, \ldots, \mathbf{u}_{m-1}$. So the number of possible points $\mathbf{u}_m$, when $\mathbf{u}_1, \ldots, \mathbf{u}_{m-1}$ are fixed, is at most

$$\left[1+\frac{2R}{\omega}\right]^{m-1}.$$

Hence
$$N \leqslant \left[1+\frac{2R}{\omega}\right]^{(n-1)(m-1)} \left[1+\frac{2R}{\omega}\right]^{m-1} = \left[1+\frac{2R}{\omega}\right]^{(m-1)n}.$$

This gives the required result.

LEMMA 2. *Suppose that* $a_1, \ldots, a_m, q$ *are positive with*

$$a_1 \geqslant a_2 \geqslant \ldots \geqslant a_m,$$

that $b_1, \ldots, b_m$ *are integers with* $(b_1, \ldots, b_m) = 1$ *and that*

$$\alpha_i \quad (1 \leqslant i \leqslant m),$$

and
$$\alpha_{ij} \quad (1 \leqslant i < j \leqslant m)$$

are real numbers. Then the number of sets of integers $u_1, \ldots, u_m$ *satisfying the inequalities*

$$\alpha_j \leqslant u_j + \sum_{i<j} \alpha_{ij} u_i \leqslant \alpha_j + a_j \quad (j = 1, \ldots, m)$$

and the congruence
$$\sum_{i=1}^{m} b_i u_i \equiv 0 \pmod{q}$$

is at most
$$(1+a_1)(1+a_2)\ldots(1+a_{m-1})\left(1+\frac{a_m}{q}\right). \tag{35}$$

Proof. The result is trivial when $m = 1$, provided (b_1) is interpreted as $|b_1|$. Suppose that the result is true when m has any value less than the value under consideration. Write $(b_m, q) = d$. Then $u_1, \ldots, u_{m-1}$ satisfy the congruence

$$\sum_{i=1}^{m-1} b_i u_i \equiv 0 \pmod{d}.$$

If $b = (b_1, \ldots, b_{m-1})$ is greater than 1, then b has no factor in common with d (such a factor would be common to $b_1, \ldots, b_m$), and so the factor b can be removed from the congruence. Thus, using the result with m replaced by $m-1$, the total number of possibilities for $u_1, \ldots, u_{m-1}$ is at most

$$(1+a_1)(1+a_2)\ldots\left(1+\frac{a_{m-1}}{d}\right).$$

Now when $u_1, \ldots, u_{m-1}$ are fixed, the inequality restricts u_m to an interval of length a_m, while the congruence determines $u_m \bmod q/d$. So for fixed $u_1, \ldots, u_{m-1}$ the number of possibilities for u_m is at most

$$1+\frac{a_m}{q/d}.$$

Hence the total number of possibilities for $u_1, \ldots, u_m$ is at most

$$(1+a_1)(1+a_2)\ldots(1+a_{m-2})\left(1+\frac{a_{m-1}}{d}\right)\left(1+\frac{a_m}{q/d}\right).$$

But
$$\left(1+\frac{a_{m-1}}{d}\right)\left(1+\frac{a_m}{q/d}\right) = (1+a_{m-1})\left(1+\frac{a_m}{q}\right)-(d-1)\left(\frac{a_{m-1}}{d}-\frac{a_m}{q}\right)$$
$$\leqslant (1+a_{m-1})\left(1+\frac{a_m}{q}\right),$$

since $a_{m-1} \geqslant a_m$ and $1 \leqslant d \leqslant q$. Thus the number of possibilities is at most the number (35), and so the result follows by induction.

5. The object of this section is to show that, as ω tends to zero, the sums J and $J(\nu;\,\mu;\,C)$ tend to certain integrals. As in § 3 we confine our attention in the first place to the case when the division $(\nu;\,\mu)$ is given by
$$\nu_1 = 1, \ldots, \nu_m = m,$$
$$\mu_1 = m+1, \ldots, \mu_{k-m} = k.$$

We have, if ω is sufficiently small,

$$J(\nu;\,\mu;\,C) = \Sigma\left[\!\!\left[\begin{matrix}\dim\{\mathbf{u}_1, \ldots, \mathbf{u}_m\} = m, \\ \sum_{i=1}^{m} c_{ij}\,\mathbf{u}_i \in L \quad (j = 1, \ldots, k-m)\end{matrix}\right]\!\!\right] I(\mathbf{u}_1, \ldots, \mathbf{u}_k),$$

where
$$\mathbf{u}_{m+s} = \sum_{i=1}^{m} c_{is}\,\mathbf{u}_i \quad (s = 1, \ldots, k-m),$$

and
$$I(\mathbf{u}_1, \ldots, \mathbf{u}_k) = \frac{N(C)}{q^m}\,\omega^{(n-1)m}\int_{-\infty}^{\infty}\!\!\ldots\!\int_{-\infty}^{\infty}\prod_{r=1}^{m}\rho_r(\omega\mathbf{u}_r, \xi_r)\prod_{s=1}^{k-m}\rho_{m+s}\left(\omega\sum_{i=1}^{m}c_{is}\,\mathbf{u}_i,\ \sum_{i=1}^{m}c_{is}\,\xi_i\right)\mathrm{d}\xi_1\ldots\mathrm{d}\xi_m.$$

Here $\mathbf{u}_{m+1}, \ldots, \mathbf{u}_k$ are unnecessary; it is convenient to write

$$F(\mathbf{x}_1, \ldots, \mathbf{x}_m) = \frac{N(C)}{q^m}\int_{-\infty}^{\infty}\!\!\ldots\!\int_{-\infty}^{\infty}\prod_{r=1}^{m}\rho_r(\mathbf{x}_r, \xi_r)\prod_{s=1}^{k-m}\rho_{m+s}\left(\sum_{i=1}^{m}c_{is}\,\mathbf{x}_i,\ \sum_{i=1}^{m}c_{is}\,\xi_i\right)\mathrm{d}\xi_1\ldots\mathrm{d}\xi_m,$$

so that
$$J(\nu;\,\mu;\,C) = \Sigma\left[\!\!\left[\begin{matrix}\dim\{\mathbf{u}_1, \ldots, \mathbf{u}_m\} = m, \\ \sum_{i=1}^{m} c_{ij}\,\mathbf{u}_i \in L \quad (j = 1, \ldots, k-m)\end{matrix}\right]\!\!\right]\omega^{(n-1)m}F(\omega\mathbf{u}_1, \ldots, \omega\mathbf{u}_m).$$

Now we have assumed that the functions $\rho_1(\mathbf{X}), \ldots, \rho_k(\mathbf{X})$ are bounded for all $\mathbf{X}$, continuous except perhaps for $\mathbf{X} = \mathbf{O}$, and vanish outside the region given by

$$|\mathbf{X}| \leqslant R.$$

Hence the function $F(\mathbf{x}_1, \ldots, \mathbf{x}_m)$ is a bounded function, continuous in the $m(n-1)$ coordinates of the points $\mathbf{x}_1, \ldots, \mathbf{x}_m$, except perhaps at the points of, at most, k linear subspaces of dimension less than $m(n-1)$, and which vanishes outside the region given by

$$|\mathbf{x}_1| \leqslant R, \ldots, |\mathbf{x}_m| \leqslant R.$$

It follows, in particular, that $F(\mathbf{x}_1, \ldots, \mathbf{x}_m)$ is integrable in the Riemann sense over the whole space of dimension $m(n-1)$.

So, by lemma 1, we have

$$\Sigma\left[\!\!\left[\begin{matrix}\dim\{\mathbf{u}_1, \ldots, \mathbf{u}_m\} < m, \\ \sum_{i=1}^{m} c_{ij}\,\mathbf{u}_i \in L \quad (j = 1, \ldots, k-m)\end{matrix}\right]\!\!\right] \omega^{(n-1)m} F(\omega\mathbf{u}_1, \ldots, \omega\mathbf{u}_m)$$
$$= O\left(\Sigma\left[\!\!\left[\begin{matrix}\dim\{\mathbf{u}_1, \ldots, \mathbf{u}_m\} < m, \\ |\mathbf{u}_1| \leqslant R/\omega, \ldots, |\mathbf{u}_m| \leqslant R/\omega\end{matrix}\right]\!\!\right] \omega^{(n-1)m}\right)$$
$$= O(\omega^{-(m-1)n+(n-1)m}) = O(\omega^{n-m}),$$

provided ω is sufficiently small. Thus

$$J(\nu;\, \mu;\, C) = o(1) + \Sigma\left[\!\!\left[\begin{matrix}\mathbf{u}_1 \in L, \ldots, \mathbf{u}_m \in L, \\ \sum_{i=1}^{m} c_{ij}\,\mathbf{u}_i \in L \quad (j = 1, \ldots, k-m)\end{matrix}\right]\!\!\right] \omega^{(n-1)m} F(\omega\mathbf{u}_1, \ldots, \omega\mathbf{u}_m) \tag{36}$$

as $\omega \to 0$.

Let us regard the set $(\mathbf{u}_1, \ldots, \mathbf{u}_m)$ of points $\mathbf{u}_1, \ldots, \mathbf{u}_m$ as a point in the space of $(n-1)\,m$ dimensions with co-ordinates

$$(u_1^{(1)}, \ldots, u_{n-1}^{(1)}, u_1^{(2)}, \ldots, u_{n-1}^{(m)}).$$

Then the conditions

$$\left.\begin{matrix}\mathbf{u}_1 \in \mathrm{L}, \ldots, \mathbf{u}_m \in L, \\ \sum_{i=1}^{m} c_{ij}\,\mathbf{u}_i \in L \quad (j = 1, \ldots, k-m),\end{matrix}\right\} \tag{37}$$

restrict the point $(\mathbf{u}_1, \ldots, \mathbf{u}_m)$ to lie on a certain sublattice of the lattice of points with integral co-ordinates. Let D be the determinant of this sublattice. Now $F(\mathbf{x}_1, \ldots, \mathbf{x}_m)$ is a function which is integrable in the Riemann sense over the whole of this space. So, by the theory of Riemann integration,

$$\Sigma\left[\!\!\left[\begin{matrix}\mathbf{u}_1 \in L, \ldots, \mathbf{u}_m \in L, \\ \sum_{i=1}^{m} c_{ij}\,\mathbf{u}_i \in L \quad (j = 1, \ldots, k-m)\end{matrix}\right]\!\!\right] \omega^{(n-1)m} F(\omega\mathbf{u}_1, \ldots, \omega\mathbf{u}_m)$$
$$\to \frac{1}{D}\int \ldots \int F(\mathbf{x}_1, \ldots, \mathbf{x}_m)\, \mathrm{d}\mathbf{x}_1 \ldots \mathrm{d}\mathbf{x}_m \tag{38}$$

as $\omega \to 0$, the integration being over the whole space.

Now the conditions (37) can be split up into the equivalent system of the conditions that, for $t = 1, \ldots, n-1$, the co-ordinates

$$u_t^{(1)}, \ldots, u_t^{(m)}$$

are integers and the numbers

$$\sum_{i=1}^{m} c_{is}\, u_t^{(i)} \quad (s = 1, \ldots, k-m)$$

are integral. Thus the sublattice of determinant D is the Cartesian product of $(n-1)$ sublattices of the lattice of points with integral co-ordinates in m-dimensional space. Each of these sublattices has determinant

$$d = q^m/N,$$

where N is the number of sets of integers $u_1, \ldots, u_m$ with

$$0 \leqslant u_r < q \quad (r = 1, \ldots, m),$$

for which the numbers

$$\sum_{i=1}^{m} c_{is}\, u_i \quad (s = 1, \ldots, k-m)$$

are integral. Hence $N = N(C)$ and

$$D = d^{n-1} = \{q^m/N(C)\}^{n-1}. \tag{39}$$

It follows from (36), (38) and (39) and the definition of $F(\mathbf{x}_1, \ldots, \mathbf{x}_m)$ that, as $\omega \to 0$,

$$J(\nu;\,\mu;\,C) \to \left\{\frac{N(C)}{q^m}\right\}^{n-1} \int \ldots \int F(\mathbf{x}_1, \ldots, \mathbf{x}_m)\, d\mathbf{x}_1 \ldots d\mathbf{x}_m$$

$$= \left\{\frac{N(C)}{q^m}\right\}^{n} \int \ldots \int \left[\int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \prod_{r=1}^{m} \rho_r(\mathbf{x}_r, \xi_r) \prod_{s=1}^{k-m} \rho_{m+s}\left(\sum_{i=1}^{m} c_{is}\mathbf{x}_i, \sum_{i=1}^{m} c_{is}\xi_i\right) d\xi_1 \ldots d\xi_m\right] d\mathbf{x}_1 \ldots d\mathbf{x}_m$$

$$= \left\{\frac{N(C)}{q^m}\right\}^{n} \int \ldots \int \prod_{r=1}^{m} \rho_r(\mathbf{X}_r) \prod_{s=1}^{k-m} \rho_{m+s}\left(\sum_{i=1}^{m} c_{is}\mathbf{X}_i\right) d\mathbf{X}_1 \ldots d\mathbf{X}_m.$$

For the case of a general division $(\nu;\,\mu)$, we obtain the corresponding result that

$$J(\nu;\,\mu;\,C) \to \left\{\frac{N(C)}{q^m}\right\}^{n} \int \ldots \int \prod_{i=1}^{m} \rho_{\nu_i}(\mathbf{X}_i) \prod_{j=1}^{k-m} \rho_{\mu_j}\left(\sum_{i=1}^{m} c_{ij}\mathbf{X}_i\right) d\mathbf{X}_1 \ldots d\mathbf{X}_m, \tag{40}$$

as $\omega \to 0$. Further, the same method shows that

$$J \to \int \ldots \int \prod_{r=1}^{k} \rho_r(\mathbf{X}_r)\, d\mathbf{X}_1 \ldots d\mathbf{X}_k = \prod_{r=1}^{k} \left\{\int \rho_r(\mathbf{X})\, d\mathbf{X}\right\}, \tag{41}$$

as $\omega \to 0$.

6. It is clear from the results (8), (40) and (41) that some progress has been made towards a proof of a result similar to theorem 1; but it is also not surprising that the rest of the proof should depend on an investigation of the uniformity of the convergence of the series (8). The object of this section is to obtain a bound for $J(\nu;\,\mu;\,C)$. As in §§ 3 and 5, we confine our attention in the first place to the case when the division $(\nu;\,\mu)$ is given by

$$\nu_1 = 1, \ldots, \nu_m = m,$$
$$\mu_1 = m+1, \ldots, \mu_{k-m} = k.$$

We have, if $0 < \omega < \omega_0$, where ω_0 depends only on n, k and R,

$$J(\nu;\,\mu;\,C) = \Sigma \left[\!\left[\begin{matrix} \dim\{\mathbf{u}_1, \ldots, \mathbf{u}_m\} = m, \\ \sum_{i=1}^{m} c_{ij}\mathbf{u}_i \in L \quad (j = 1, \ldots, k-m) \end{matrix} \right]\!\right] I(\mathbf{u}_1, \ldots, \mathbf{u}_k),$$

where
$$\mathbf{u}_{m+s} = \sum_{i=1}^{m} c_{is}\mathbf{u}_i \quad (s = 1, \ldots, k-m),$$

and

$$I(\mathbf{u}_1, \ldots, \mathbf{u}_k) = \frac{N(C)}{q^m}\, \omega^{(n-1)m} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \prod_{r=1}^{m} \rho_r(\omega\mathbf{u}_r, \xi_r) \prod_{s=1}^{k-m} \rho_{m+s}\left(\omega \sum_{i=1}^{m} c_{is}\mathbf{u}_i, \sum_{i=1}^{m} c_{is}\xi_i\right) d\xi_1 \ldots d\xi_m.$$

Now
$$I(\mathbf{u}_1, \ldots, \mathbf{u}_k) = 0$$
unless
$$|\mathbf{u}_r| \leqslant R/\omega \quad (r = 1, \ldots, m),$$
$$\left|\sum_{i=1}^{m} c_{is}\mathbf{u}_i\right| \leqslant R/\omega \quad (s = 1, \ldots, k-m),$$
and in any case

$$I(\mathbf{u}_1, \ldots, \mathbf{u}_k) = O\left(\frac{N(C)}{q^m}\, \omega^{(n-1)m} \int_{-R}^{R} \ldots \int_{-R}^{R} \left[\!\left[\left|\sum_{i=1}^{m} c_{is}\xi_i\right| < R \quad (s = 1, \ldots, k-m) \right]\!\right] d\xi_1 \ldots d\xi_m\right),$$

where the constant implied by the O-notation depends only on n, k, R and the maximum attained by the moduli of the functions $\rho_1(\mathbf{X}), \ldots, \rho_k(\mathbf{X})$. Write

$$c = \max_{\substack{i=1,\ldots,m, \\ j=1,\ldots,k-m}} |c_{ij}|.$$

We may suppose without loss of generality that

$$c_{11} = c.$$

Then the conditions

$$\left|\sum_{i=1}^{m} c_{is}\xi_i\right| < R \quad (s = 1, \ldots, k-m)$$

imply that

$$\left|\xi_1 + \sum_{i=2}^{m} \frac{c_{i1}}{c}\xi_i\right| \leqslant \frac{R}{c}.$$

So, for fixed $\xi_2, \ldots, \xi_m$, we have

$$\int_{-R}^{R} \left[\!\!\left[\left|\sum_{i=1}^{m} c_{is}\xi_i\right| < R \quad (s = 1, \ldots, k-m) \right]\!\!\right] d\xi_1 \leqslant 2R \min\left[1, \frac{1}{c}\right].$$

Thus

$$I(\mathbf{u}_1, \ldots, \mathbf{u}_k) = O\left(\frac{N(C)}{q^m}\omega^{(n-1)m} \min\left[1, \frac{1}{c}\right]\right).$$

This result shows that

$$J(\nu; \mu; C) = O\left(\Sigma \left[\!\!\left[\begin{array}{c} \dim\{\mathbf{u}_1, \ldots, \mathbf{u}_m\} = m, \\ |\mathbf{u}_1| \leqslant R/\omega, \ldots, |\mathbf{u}_m| \leqslant R/\omega, \\ \sum_{i=1}^{m} c_{ij}\mathbf{u}_i \in L \quad (j = 1, \ldots, k-m), \\ \left|\sum_{i=1}^{m} c_{ij}\mathbf{u}_i\right| \leqslant R/\omega \quad (j = 1, \ldots, k-m) \end{array} \right]\!\!\right] \frac{N(C)}{q^m}\omega^{(n-1)m} \min\left[1, \frac{1}{c}\right]\right)$$

$$= O\left(\frac{N(C)}{q^m}\omega^{(n-1)m} \min\left[1, \frac{1}{c}\right] \Sigma \left[\!\!\left[\begin{array}{c} \mathbf{u}_1 \in L, \ldots, \mathbf{u}_m \in L, \\ |\mathbf{u}_1| \leqslant R/\omega, \ldots, |\mathbf{u}_m| \leqslant R/\omega, \\ \sum_{i=1}^{m} c_{ij}\mathbf{u}_i \in L \quad (j = 1, \ldots, k-m), \\ \left|\sum_{i=1}^{m} c_{ij}\mathbf{u}_i\right| \leqslant R/\omega \quad (j = 1, \ldots, k-m) \end{array} \right]\!\!\right]_1 \right). \tag{42}$$

But

$$\Sigma \left[\!\!\left[\begin{array}{c} \mathbf{u}_1 \in L, \ldots, \mathbf{u}_m \in L, \\ |\mathbf{u}_1| \leqslant R/\omega, \ldots, |\mathbf{u}_m| \leqslant R/\omega, \\ \sum_{i=1}^{m} c_{ij}\mathbf{u}_i \in L \quad (j = 1, \ldots, k-m), \\ \left|\sum_{i=1}^{m} c_{ij}\mathbf{u}_i\right| \leqslant R/\omega \quad (j = 1, \ldots, k-m) \end{array} \right]\!\!\right]_1$$

$$= \prod_{t=1}^{n-1} \left(\sum_{u_t^{(1)}, \ldots, u_t^{(m)}} \left[\!\!\left[\begin{array}{c} |u_t^{(1)}| \leqslant R/\omega, \ldots, |u_t^{(m)}| \leqslant R/\omega, \\ \sum_{i=1}^{m} c_{ij}u_t^{(i)} \equiv 0 \pmod 1 \quad (j = 1, \ldots, k-m), \\ \left|\sum_{i=1}^{m} c_{ij}u_t^{(i)}\right| \leqslant R/\omega \quad (j = 1, \ldots, k-m) \end{array} \right]\!\!\right]_1 \right)$$

$$= \left(\sum_{u_1, \ldots, u_m} \left[\!\!\left[\begin{array}{c} |u_1| \leqslant R/\omega, \ldots, |u_m| \leqslant R/\omega, \\ \sum_{i=1}^{m} c_{ij}u_i \equiv 0 \pmod 1 \quad (j = 1, \ldots, k-m), \\ \left|\sum_{i=1}^{m} c_{ij}u_i\right| \leqslant R/\omega \quad (j = 1, \ldots, k-m) \end{array} \right]\!\!\right]_1 \right)^{n-1}$$

$$= N^{n-1}, \tag{43}$$

where N is the number of sets of integers $u_1, \ldots, u_m$ satisfying the conditions

$$\left.\begin{aligned} &|u_1| \leqslant R/\omega, \ldots, |u_m| \leqslant R/\omega, \\ &\sum_{i=1}^{m} c_{ij} u_i \equiv 0 \pmod 1 \quad (j = 1, \ldots, k-m), \\ &\left|\sum_{i=1}^{m} c_{ij} u_i\right| \leqslant R/\omega \quad (j = 1, \ldots, k-m). \end{aligned}\right\} \tag{44}$$

It is convenient to write

$$q_{is} = qc_{is} \quad (i = 1, \ldots, m;\ s = 1, \ldots, k-m).$$

Then, since the numbers c_{is} are rational numbers with least common denominator q, it is clear that the numbers q and q_{is} $(i = 1, \ldots, m; s = 1, \ldots, k-m)$ are integers with no common divisor greater than 1. It follows from the theory of elementary divisors* that there are integers $\lambda_0, \lambda_1, \ldots, \lambda_{k-m}$ such that the integers

$$b_i = \lambda_0 q + \sum_{s=1}^{k-m} \lambda_s q_{is} \quad (i = 1, \ldots, m)$$

have no common factor exceeding 1. Then the congruences in the conditions (44) imply that

$$\sum_{i=1}^{m} b_i u_i \equiv \sum_{i=1}^{m} \sum_{s=1}^{k-m} \lambda_s q_{is} u_i \equiv \sum_{s=1}^{k-m} \lambda_s \sum_{i=1}^{m} qc_{is} u_i \equiv 0 \pmod q.$$

Further, the inequality conditions in (44) imply that

$$|u_1| \leqslant R/\omega, \ldots, |u_m| \leqslant R/\omega,$$

$$\left|u_1 + \sum_{i=2}^{m} \frac{c_{i1}}{c} u_i\right| \leqslant R/(\omega c).$$

Consequently, by lemma 2 of § 5, the number N of sets of integers satisfying the conditions (44) satisfies

$$\begin{aligned} N &\leqslant \left(1 + \frac{2R}{\omega}\right)^{m-1} \min\left[\left(1 + \frac{2R}{\omega q}\right), \left(1 + \frac{2R}{\omega c q}\right)\right] \\ &= O\left(\omega^{-m} \min\left[\omega + \frac{1}{q}, \omega + \frac{1}{cq}\right]\right). \end{aligned} \tag{45}$$

It is convenient at this stage to note that, if l is a large positive integer, and $a_1, \ldots, a_m$ are integers satisfying

$$0 \leqslant a_r < lq \quad (r = 1, \ldots, m)$$

for which the numbers

$$\sum_{i=1}^{m} c_{is} a_i \quad (s = 1, \ldots, k-m)$$

are integral, then these integers satisfy

$$0 \leqslant a_r \leqslant lq - 1 \quad (r = 1, \ldots, m),$$

$$\sum_{i=1}^{m} b_i a_i \equiv 0 \pmod q.$$

* See, for example, the lemma on p. 635 of Mahler (1949).

It follows from lemma 2, that the number $l^m N(C)$ of such sets of integers satisfies

$$l^m N(C) \leqslant (lq)^{m-1}\left(1+\frac{lq-1}{q}\right).$$

As l may be taken to be arbitrarily large, it follows that

$$N(C) \leqslant q^{m-1}. \tag{46}$$

Combining the results (42), (43), (45) and (46), we obtain the result that

$$\begin{aligned} J(\nu;\,\mu;\,C) &= O\left(\frac{1}{q}\,\omega^{(n-1)m}\min\left[1,\frac{1}{c}\right]\left\{\omega^{-m}\min\left[\omega+\frac{1}{q},\omega+\frac{1}{cq}\right]\right\}^{n-1}\right)\\ &= O\left(\frac{1}{q}\min\left[1,\frac{1}{c}\right]\left\{\min\left[\omega+\frac{1}{q},\omega+\frac{1}{cq}\right]\right\}^{n-1}\right). \end{aligned}$$

But, by (29) and (31), it is clear that $I(\mathbf{u}_1, \ldots, \mathbf{u}_k)$ and consequently $J(\nu;\,\mu;\,C)$ vanishes unless

$$q \leqslant m^{\frac{1}{2}m}(R/\omega)^m$$

and

$$qc \leqslant m^{\frac{1}{2}m}(R/\omega)^m,$$

i.e. unless

$$\omega \leqslant R\sqrt{m}\min\left[\frac{1}{q^{1/m}},\frac{1}{(qc)^{1/m}}\right].$$

So, in obtaining an upper bound for $J(\nu;\,\mu;\,C)$, we may suppose that this condition is satisfied. Thus

$$\begin{aligned} J(\nu;\,\mu;\,C) &= O\left(\frac{1}{q}\min\left[1,\frac{1}{c}\right]\left\{\min\left[\frac{1}{q^{1/m}}+\frac{1}{q},\frac{1}{(qc)^{1/m}}+\frac{1}{qc}\right]\right\}^{n-1}\right)\\ &= O\left(\frac{1}{q}\min\left[1,\frac{1}{c}\right]\left\{\min\left[\frac{1}{q^{1/m}},\frac{1}{(qc)^{1/m}}\right]\right\}^{n-1}\right)\\ &= O\left(\left(\frac{1}{q}\right)^{1+\{(n-1)/m\}}\min\left[1,\left(\frac{1}{c}\right)^{1+\{(n-1)/m\}}\right]\right). \end{aligned} \tag{47}$$

It is easy to check that this estimate remains valid in the case of a general division $(\nu;\,\mu)$. It should be remembered that this estimate is valid provided $\omega<\omega_0$, where $\omega_0>0$ and depends only on n, k and R; while the constant implied by the O-notation depends only on n, k, R and the maximum modulus of the functions $\rho_1(\mathbf{X}), \ldots, \rho_k(\mathbf{X})$.

7. In this section the estimate (47) is used to show that the sum on the right of the formula

$$\int_0^1 \cdots \int_0^1 \prod_{r=1}^{k} \rho_r(\Lambda(\alpha_1, \ldots, \alpha_{n-1}, \omega))\,d\alpha_1 \ldots d\alpha_{n-1} = \left\{\prod_{r=1}^{k}\rho_r(\mathbf{O})\right\} + J + \sum_{(\nu;\,\mu)}\sum_{q=1}^{\infty}\sum_{C} J(\nu;\,\mu;\,C), \tag{48}$$

proved in § 2, is uniformly convergent for $0<\omega<\omega_0$, provided

$$\max_{m=1,\ldots,k}\,[m^2(k-m)+1] < n. \tag{49}$$

Since, for each value of m with $1 \leqslant m \leqslant k-1$, there are only a finite number of divisions $(\nu;\,\mu)$, it suffices to prove the uniform convergence of the sum

$$\sum_{q=1}^{\infty}\sum_{C} J(\nu;\,\mu;\,C)$$

for each division $(\nu;\,\mu)$.

Since ω does not appear in the estimate (47), valid for $0<\omega<\omega_0$, and the constant implied by the O-notation is independent of ω, it clearly suffices to prove the convergence of the sum

$$\sum_{q=1}^{\infty}\sum_{C} M(C), \tag{50}$$

where
$$M(C)=\left(\frac{1}{q}\right)^{1+\{(n-1)/m\}}\min\left[1,\left(\frac{1}{c}\right)^{1+\{(n-1)/m\}}\right].$$

For a fixed q the number of matrices C with $c\leqslant 1$ is at most of the order

$$O(q^{m(k-m)}).$$

So
$$\sum_{\substack{C\\ c\leqslant 1}} M(C)=O(q^{m(k-m)}\,q^{-1-\{(n-1)/m\}})$$
$$=O(q^{-1-[n-1-m^2(k-m)]/m}),$$

and
$$\sum_{q=1}^{\infty}\sum_{\substack{C\\ c\leqslant 1}} M(C) \tag{51}$$

is convergent provided
$$m^2(k-m)+1<n. \tag{52}$$

For a fixed q and a fixed integer l with $l>q$, the number of matrices C with $c=l/q$ is at most of order
$$O((cq)^{m(k-m)-1}).$$

So
$$\sum_{\substack{C\\ c=l/q}} M(C)=O((cq)^{m(k-m)-1}(cq)^{-1-\{(n-1)/m\}})$$
$$=O(l^{-2-[n-1-m^2(k-m)]/m}).$$

Thus, provided the condition (52) is satisfied,

$$\sum_{l=q+1}^{\infty}\sum_{\substack{C\\ c=l/q}} M(C)=O(q^{-1-[n-1-m^2(k-m)]/m}),$$

and the sum
$$\sum_{q=1}^{\infty}\sum_{\substack{C\\ c>1}} M(C)$$

is convergent.

Combining this result with the result that the sum (51) is convergent, we see that the sum (50) is convergent. This completes the proof that the sum on the right-hand side of (48) converges uniformly for $0<\omega<\omega_0$, provided the condition (49) is satisfied.

8. This section proves a theorem that clearly contains theorem 1 as a special case.

THEOREM 2. *Let $\rho_1(\mathbf{X}),\ldots,\rho_k(\mathbf{X})$ be functions which are integrable in the Riemann sense over the whole of n-dimensional space. Suppose that k satisfies the condition*

$$\max_{m=1,\ldots,k}[m^2(k-m)+1]<n. \tag{53}$$

Then, provided the integral is interpreted as an upper Riemann integral, the mean value

$$\int_0^1\cdots\int_0^1\prod_{r=1}^{k}\rho_r(\Lambda(\alpha_1,\ldots,\alpha_{n-1},\omega))\,\mathrm{d}\alpha_1\ldots\mathrm{d}\alpha_{n-1} \tag{54}$$

exists, and, as $\omega\to 0$, tends to the limit

$$\prod_{r=1}^{k}\rho_r(\mathbf{O})+\prod_{r=1}^{k}\int\rho_r(\mathbf{X})\,\mathrm{d}\mathbf{X}+\sum_{(\mu;\nu)}\sum_{q=1}^{\infty}\sum_{C}\left(\frac{N(C)}{q^m}\right)^n\int\cdots\int\prod_{i=1}^{m}\rho_{\nu_i}(\mathbf{X}_i)\prod_{j=1}^{k-m}\rho_{\mu_j}\left(\sum_{i=1}^{m}c_{ij}\mathbf{X}_i\right)\mathrm{d}\mathbf{X}_1\ldots\mathrm{d}\mathbf{X}_m, \tag{55}$$

where the outer sum is over all divisions $(\nu;\mu) = (\nu_1,\dots,\nu_m;\mu_1,\dots,\mu_{k-m})$ *of the numbers* $1, 2, \dots, k$ *into two sequences* $\nu_1,\dots,\nu_m$ *and* $\mu_1,\dots,\mu_{k-m}$, *with* $1\leqslant m\leqslant k-1$,

$$\left.\begin{aligned}&1\leqslant \nu_1<\nu_2<\dots<\nu_m\leqslant k,\\&1\leqslant \mu_1<\mu_2<\dots<\mu_{k-m}\leqslant k,\\&\nu_i\neq\mu_j \quad \textit{if} \quad 1\leqslant i\leqslant m, \quad 1\leqslant j\leqslant k-m,\end{aligned}\right\} \tag{56}$$

where the inner sum is over all $m\times(k-m)$ *matrices* C *with rational elements with lowest common denominator* q *and with*

$$c_{is} = 0 \quad \textit{if} \quad \mu_s<\nu_i, \tag{57}$$

and where $N(C)$ *is the number of sets of integers* $a_1,\dots,a_m$ *satisfying*

$$0\leqslant a_r<q \quad (r = 1,\dots,m), \tag{58}$$

for which the numbers

$$\sum_{i=1}^{m} c_{is}a_i \quad (s = 1,\dots,k-m), \tag{59}$$

are integral.

Proof. First consider the case when $\rho_1(\mathbf{X}),\dots,\rho_k(\mathbf{X})$ are also assumed to be continuous, except perhaps at $\mathbf{X} = \mathbf{O}$. Then, by the work of the preceding sections, the integral (54) exists for all ω with $\omega>0$ and has the value

$$\prod_{r=1}^{k}\rho_r(\mathbf{O})+J+\sum_{(\nu;\mu)}\sum_{q=1}^{\infty}\sum_{C}J(\nu;\mu;C),$$

provided $\omega<\omega_0$. Further, this sum is uniformly convergent for $0<\omega<\omega_0$. Again, as $\omega\to 0$, we have

$$J\to\prod_{r=1}^{k}\int\rho_r(\mathbf{X})\,\mathrm{d}\mathbf{X},$$

$$J(\nu;\mu;C)\to\left(\frac{N(C)}{q^m}\right)^n\int\dots\int\prod_{i=1}^{m}\rho_{\nu_i}(\mathbf{X}_i)\prod_{j=1}^{k-m}\rho_{\mu_j}\left(\sum_{i=1}^{m}c_{ij}\mathbf{X}_i\right)\mathrm{d}\mathbf{X}_1\dots\mathrm{d}\mathbf{X}_m.$$

So it follows, in this case, that the integral (54) tends to the value (55) as $\omega\to 0$.

Now consider the case when the functions $\rho_1(\mathbf{X}),\dots,\rho_k(\mathbf{X})$ are Riemann integrable and non-negative. Then it is possible, for every positive integer l, to find functions $\sigma_1^{(l)}(\mathbf{X}),\dots,\sigma_k^{(l)}(\mathbf{X})$ which are bounded, continuous except perhaps at $\mathbf{X} = \mathbf{O}$, which vanish outside a bounded region and which satisfy

$$\sigma_r^{(1)}(\mathbf{X})\geqslant\sigma_r^{(2)}(\mathbf{X})\geqslant\dots\geqslant\rho_r(\mathbf{X}) \quad (r = 1,\dots,k), \tag{60}$$

for all $\mathbf{X}$,

$$\sigma_r^{(1)}(\mathbf{O}) = \sigma_r^{(2)}(\mathbf{O}) = \dots = \rho_r(\mathbf{O}) \quad (r = 1,\dots,k), \tag{61}$$

and

$$\int\sigma_r^{(l)}(\mathbf{X})\,\mathrm{d}\mathbf{X}\to\int\rho_r(\mathbf{X})\,\mathrm{d}\mathbf{X} \quad (r = 1,\dots,k), \tag{62}$$

as $l\to\infty$. Then, using the result of the last paragraph, for each positive integer l,

$$\begin{aligned}\limsup_{\omega\to+0}&\int_0^1\dots\int_0^1\prod_{r=1}^{k}\rho_r(\Lambda(\alpha_1,\dots,\alpha_{n-1},\omega))\,\mathrm{d}\alpha_1\dots d\alpha_{n-1}\\&\leqslant\lim_{\omega\to+0}\int_0^1\dots\int_0^1\prod_{r=1}^{k}\sigma_r^{(l)}(\Lambda(\alpha_1,\dots,\alpha_{n-1},\omega))\,\mathrm{d}\alpha_1\dots\mathrm{d}\alpha_{n-1}\\&=\prod_{r=1}^{k}\sigma_r^{(l)}(\mathbf{O})+\prod_{r=1}^{k}\int\sigma_r^{(l)}(\mathbf{X})\,\mathrm{d}\mathbf{X}\\&\quad+\sum_{(\nu;\mu)}\sum_{q=1}^{\infty}\sum_{C}\left(\frac{N(C)}{q^m}\right)^n\int\dots\int\prod_{i=1}^{m}\sigma_{\nu_i}^{(l)}(\mathbf{X}_i)\prod_{j=1}^{k-m}\sigma_{\mu_j}^{(l)}\left(\sum_{i=1}^{m}c_{ij}\mathbf{X}_i\right)\mathrm{d}\mathbf{X}_1\dots\mathrm{d}\mathbf{X}_m.\end{aligned} \tag{63}$$

Note that (60) implies that this sum will be uniformly convergent for the different values of l. We shall show that (60), (61) and (62) imply that, as l tends to infinity, each term tends to the corresponding term of the sum (55). It is clearly sufficient to consider a term corresponding to a division $(\nu; \mu) = (1, \ldots, m; m+1, \ldots, k)$. Then the difference between corresponding terms will be a constant multiple of

$$\int \cdots \int \prod_{r=1}^{m} \sigma_r^{(l)}(\mathbf{X}_r) \prod_{s=1}^{k-m} \sigma_{m+s}^{(l)}\left(\sum_{i=1}^{m} c_{is}\mathbf{X}_i\right) d\mathbf{X}_1 \ldots d\mathbf{X}_m$$
$$-\int \cdots \int \prod_{r=1}^{m} \rho_r(\mathbf{X}_r) \prod_{s=1}^{k-m} \rho_{m+s}\left(\sum_{i=1}^{m} c_{is}\mathbf{X}_i\right) d\mathbf{X}_1 \ldots d\mathbf{X}_m$$
$$= \sum_{t=1}^{k} \int \cdots \int \rho_1 \rho_2 \ldots \rho_{t-1} \{\sigma_t^{(l)} - \rho_t\} \sigma_{t+1}^{(l)} \sigma_{t+2}^{(l)} \ldots \sigma_k^{(l)} \, d\mathbf{X}_1 \ldots d\mathbf{X}_m,$$

where we have omitted the arguments from the functions for simplicity. Now, if $1 \leqslant t \leqslant m$, we have

$$0 \leqslant \int \cdots \int \rho_1 \rho_2 \ldots \rho_{t-1} \{\sigma_t^{(l)} - \rho_t\} \sigma_{t+1}^{(l)} \sigma_{t+2}^{(l)} \ldots \sigma_k^{(l)} \, d\mathbf{X}_1 \ldots d\mathbf{X}_m$$
$$\leqslant \left(\prod_{r=1}^{t-1} \int \rho_r \, d\mathbf{X}_r\right) \left(\int \sigma_t^{(l)}(\mathbf{X}) \, d\mathbf{X} - \int \rho_t(\mathbf{X}) \, d\mathbf{X}\right) \left(\prod_{r=t+1}^{m} \int \sigma_r^{(1)} \, d\mathbf{X}_r\right) \sigma^{k-m},$$

where σ is an upper bound for the functions $\sigma_{m+1}, \ldots, \sigma_k$. In the special case when $m < t \leqslant k$ and

$$c_{1(t-m)} = c_{2(t-m)} = \ldots = c_{m(t-m)} = 0,$$

we have, by (61),

$$\int \cdots \int \rho_1 \rho_2 \ldots \rho_{t-1} \{\sigma_t^{(l)} - \rho_t\} \sigma_{t+1}^{(l)} \sigma_{t+2}^{(l)} \ldots \sigma_k^{(l)} \, d\mathbf{X}_1 \ldots d\mathbf{X}_m$$
$$= \int \cdots \int \rho_1 \rho_2 \ldots \rho_{t-1} \{\sigma_t^{(l)}(\mathbf{O}) - \rho_t(\mathbf{O})\} \sigma_{t+1}^{(l)} \sigma_{t+2}^{(l)} \ldots \sigma_k^{(l)} \, d\mathbf{X}_1 \ldots d\mathbf{X}_m$$
$$= 0.$$

In the case when $m < t \leqslant k$ and at least one, say $c_{1(t-m)}$, of the numbers $c_{1(t-m)}, \ldots, c_{m(t-m)}$ is not zero, we have

$$0 \leqslant \int \cdots \int \rho_1 \rho_2 \ldots \rho_{t-1} \{\sigma_t^{(l)} - \rho_t\} \sigma_{t+1}^{(l)} \sigma_{t+2}^{(l)} \ldots \sigma_k^{(l)} \, d\mathbf{X}_1 \ldots d\mathbf{X}_m$$
$$\leqslant \rho \sigma^{k-m-1} \int \cdots \int \rho_2(\mathbf{X}_2) \ldots \rho_m(\mathbf{X}_m) \left\{\sigma_t^{(l)}\left(\sum_{i=1}^{m} c_{i(t-m)} \mathbf{X}_i\right) - \rho_t\left(\sum_{i=1}^{m} c_{i(t-m)} \mathbf{X}_i\right)\right\} d\mathbf{X}_1 \ldots d\mathbf{X}_m$$
$$= \rho \sigma^{k-m-1} \, |c_{1(t-m)}|^{-n} \left(\prod_{r=2}^{m} \int \rho_r(\mathbf{X}_r) \, d\mathbf{X}_r\right) \left(\int \sigma_t^{(l)}(\mathbf{X}) \, d\mathbf{X} - \int \rho_t(\mathbf{X}) \, d\mathbf{X}\right),$$

where ρ is an upper bound for the function ρ_1. Thus it follows from (62) that the integral

$$\int \cdots \int \rho_1 \rho_2 \ldots \rho_{t-1} \{\sigma_t^{(l)} - \rho_t\} \sigma_{t+1}^{(l)} \sigma_{t+2}^{(l)} \ldots \sigma_k^{(l)} \, d\mathbf{X}_1 \ldots d\mathbf{X}_m$$

tends to zero as l tends to infinity for each t with $1 \leqslant t \leqslant m$. This shows that the differences between the corresponding terms in the sums (63) and (55) tend to zero. It follows from the uniformity of the convergence of the sum (63) that this sum tends to the sum (55) as l tends to infinity. Hence the left-hand side of the inequality (63) does not exceed the sum (55).

A similar argument, based on approximation to the functions $\rho_1(\mathbf{X}), \ldots, \rho_k(\mathbf{X})$ from below, shows that

$$\liminf_{\omega\to+0} \int_0^1 \ldots \int_0^1 \prod_{r=1}^{k} \rho_r(\Lambda(\alpha_1, \ldots, \alpha_{n-1}, \omega))\, d\alpha_1 \ldots d\alpha_{n-1}$$

is not less than the sum (55). Consequently, both the limits

$$\left.\begin{array}{l} \overline{\lim_{\omega\to+0}} \int_0^1 \ldots \int_0^1 \prod_{r=1}^{k} \rho_r(\Lambda(\alpha_1, \ldots, \alpha_{n-1}, \omega))\, d\alpha_1 \ldots d\alpha_{n-1}, \\ \underline{\lim_{\omega\to+0}} \int_0^1 \ldots \int_0^1 \prod_{r=1}^{k} \rho_r(\Lambda(\alpha_1, \ldots, \alpha_{n-1}, \omega))\, d\alpha_1 \ldots d\alpha_{n-1}, \end{array}\right\} \tag{64}$$

exist and are equal to the sum (55).

Now, in the general case when $\rho_1(\mathbf{X}), \ldots, \rho_k(\mathbf{X})$ are any functions integrable in the Riemann sense over the whole of space, each function can be expressed as the difference of two non-negative Riemann-integrable functions. Applying the results of the last paragraph to the different products of these functions and combining these results with suitable signs, it is easy to see that the limit (64) will exist and have the value (55). This proves the theorem.

It is perhaps worth remarking that the above method, with a few minor modifications, suffices to prove a more general result, which can, with the notation introduced in the beginning of § 2, be stated in the following way.

THEOREM 3. *Let $\rho(\mathbf{X}_1, \ldots, \mathbf{X}_k)$ be a function, which is continuous in the nk-dimensional space of points $(\mathbf{X}_1, \ldots, \mathbf{X}_k)$, and which vanishes outside a bounded region of this space. Suppose that k satisfies the condition*

$$\max_{m=1,\ldots,k} [m^2(k-m)+1] < n. \tag{65}$$

Then the limit

$$\lim_{\omega\to+0} \int_0^1 \ldots \int_0^1 \sum_{\mathbf{U}_1\in\Lambda, \ldots, \mathbf{U}_k\in\Lambda} \rho(\alpha\mathbf{U}_1, \ldots, \alpha\mathbf{U}_k)\, d\alpha_1 \ldots d\alpha_{n-1} \tag{66}$$

exists and has the value

$$\rho(\mathbf{O}, \ldots, \mathbf{O}) + \int \ldots \int \rho(\mathbf{X}_1, \ldots, \mathbf{X}_k)\, d\mathbf{X}_1 \ldots d\mathbf{X}_k$$
$$+ \sum_{(\nu;\mu)} \sum_{q=1}^{\infty} \sum_{D} \left(\frac{N(D)}{q^m}\right)^n \int \ldots \int \rho\left(\sum_{i=1}^{m} d_{i1}\mathbf{X}_i, \ldots, \sum_{i=1}^{m} d_{ik}\mathbf{X}_i\right) d\mathbf{X}_1 \ldots d\mathbf{X}_m, \tag{67}$$

where the outer sum is over all divisions $(\nu; \mu)$ of the type described in the statements of theorems 1 and 2, where the inner sum is over all $m \times k$ matrices D with rational elements with lowest common denominator q and with

$$\left.\begin{array}{ll} d_{i\nu_j} = \delta_{ij} & (i = 1, \ldots, m;\ j = 1, \ldots, m), \\ d_{i\mu_j} = 0, \quad \text{if} \quad \mu_j < \nu_i & (i = 1, \ldots, m;\ j = 1, \ldots, k-m), \end{array}\right\} \tag{68}$$

and where N(D) is the number of sets of integers $a_1, \ldots, a_m$ satisfying

$$0 \leqslant a_r < q \quad (r = 1, \ldots, m), \tag{69}$$

for which the numbers

$$\sum_{i=1}^{m} d_{ij} a_i \quad (j = 1, \ldots, k), \tag{70}$$

are integral.

9. In this section we consider the case when

$$\rho_r(\mathbf{X}) = \rho(\mathbf{X}) \quad (r = 1, \ldots, k),$$

and $\rho(\mathbf{X})$ is non-negative. We obtain bounds for the terms in the sum (4) and for certain groups of these terms. For convenience, we suppose that

$$\rho(\mathbf{X}) \leqslant 1 \quad \text{for all } \mathbf{X},$$

and we write

$$\int \rho(\mathbf{X})\, d\mathbf{X} = V.$$

We first obtain a bound for the integral

$$I(\nu;\, \mu;\, C) = \int \ldots \int \prod_{r=1}^{m} \rho(\mathbf{X}_r) \prod_{s=1}^{k-m} \rho\left(\sum_{i=1}^{m} c_{is} \mathbf{X}_i\right) d\mathbf{X}_1 \ldots d\mathbf{X}_m.$$

Write

$$k_{ir} = \delta_{ir} \quad (i, r = 1, \ldots, m),$$

$$k_{ir} = c_{i,r-m} \quad (i = 1, \ldots, m;\ r = m+1, \ldots, k),$$

and

$$\mathbf{Y}_r = \sum_{i=1}^{m} k_{ir} \mathbf{X}_i \quad (r = 1, \ldots, k).$$

Then

$$I(\nu;\, \mu;\, C) = \int \ldots \int \prod_{r=1}^{k} \rho(\mathbf{Y}_r)\, d\mathbf{X}_1 \ldots d\mathbf{X}_m.$$

Now, if $\lambda_1, \ldots, \lambda_m$ is any selection of m distinct numbers from the numbers $1, \ldots, k$, such that the matrix

$$(k_{i\lambda}) \quad (\lambda = \lambda_1, \ldots, \lambda_m)$$

is non-singular with a determinant of absolute value D, then the equations

$$\mathbf{Z}_j = \sum_{i=1}^{m} k_{i\lambda_j} \mathbf{X}_i \quad (j = 1, \ldots, m)$$

define a linear transformation in the mn-dimensional space of points $(\mathbf{X}_1, \ldots, \mathbf{X}_m)$, with determinant $\pm D^n$. Thus

$$\begin{aligned} I(\nu;\, \mu;\, C) &= \int \ldots \int \prod_{r=1}^{k} \rho(\mathbf{Y}_r)\, d\mathbf{X}_1 \ldots d\mathbf{X}_m \\ &\leqslant \int \ldots \int \prod_{j=1}^{m} \rho(\mathbf{Z}_j)\, d\mathbf{X}_1 \ldots d\mathbf{X}_m \\ &= D^{-n} \int \ldots \int \prod_{j=1}^{m} \rho(\mathbf{Z}_j)\, d\mathbf{Z}_1 \ldots d\mathbf{Z}_m \\ &= D^{-n} V^m. \end{aligned}$$

Thus we have

$$I(\nu;\, \mu;\, C) \leqslant \{M(C)\}^{-n}\, V^m, \tag{71}$$

where $M(C)$ is the largest value taken by the absolute value of the determinant of one of the $m \times m$ minors of the matrix (k_{ir}). But as this matrix is formed by combining the $m \times m$ unit matrix with the matrix C, it is clear that $M(C)$ will be the larger of the numbers 1 and the largest value taken by the absolute value of the determinant of *any* minor of the matrix C. In particular,

$$\begin{aligned} M(C) &\geqslant \max\{1, |c_{11}|, |c_{12}|, \ldots, |c_{m,k-m}|\} \\ &= \max\{1, c\}, \end{aligned}$$

so that

$$I(\nu;\, \mu;\, C) \leqslant \min\{1, c^{-n}\}\, V^m. \tag{72}$$

We first consider the terms in the sum (4) with $q \geq 2$. For a fixed q, the number of matrices C with $c \leq 1$ is at most

$$(2q+1)^{m(k-m)} \leq (\tfrac{5}{2}q)^{m(k-m)}.$$

So, by (46) and (72),

$$\begin{aligned}\sum_{q=2}^{\infty} \sum_{\substack{C \\ c\leq 1}} \left(\frac{N(C)}{q^m}\right)^n I(\nu;\, \mu;\, C) &\leq \sum_{q=2}^{\infty} (\tfrac{5}{2}q)^{m(k-m)}\, q^{-n} V^m \\ &\leq (\tfrac{5}{2})^{m(k-m)} 2^{-n+m(k-m)+2} \sum_{q=2}^{\infty} q^{-2} V^m \\ &< 4(5)^{m(k-m)} 2^{-n} V^m, \end{aligned} \tag{73}$$

provided $$n > m(k-m)+1.$$

For a fixed $q \geq 1$ and a fixed integer l with $l > q$, the number of matrices C with $c = l/q$ is at most*

$$(2l+1)^{m(k-m)} \leq (\tfrac{5}{2}l)^{m(k-m)}.$$

So, by (46) and (72),

$$\sum_{\substack{C \\ c=l/q}} \left(\frac{N(C)}{q^m}\right)^n I(\nu;\, \mu;\, C) \leq (\tfrac{5}{2}l)^{m(k-m)}\, q^{-n} c^{-n} V^m.$$

Thus
$$\begin{aligned}\sum_{l=q+1}^{\infty} \sum_{\substack{C \\ c=l/q}} \left(\frac{N(C)}{q^m}\right)^n I(\nu;\, \mu;\, C) &\leq (\tfrac{5}{2})^{m(k-m)}\, V^m \sum_{l=q+1}^{\infty} l^{-n+m(k-m)} \\ &\leq (\tfrac{5}{2})^{m(k-m)}\, V^m (q+1)^{-n+m(k-m)+2} \sum_{l=q+1}^{\infty} \frac{1}{l(l-1)} \\ &\leq 2(\tfrac{5}{2})^{m(k-m)}\, V^m (q+1)^{-n+m(k-m)+1}, \end{aligned}$$

provided $$n \geq m(k-m)+2.$$

Hence, provided $$n > m(k-m)+2,$$

it follows that
$$\begin{aligned}\sum_{q=1}^{\infty} \sum_{\substack{C \\ c>1}} \left(\frac{N(C)}{q^m}\right)^n I(\nu;\, \mu;\, C) &\leq 2(\tfrac{5}{2})^{m(k-m)}\, V^m \sum_{q=1}^{\infty} (q+1)^{-n+m(k-m)+1} \\ &\leq 2(\tfrac{5}{2})^{m(k-m)}\, V^m 2^{-n+m(k-m)+3} \sum_{r=2}^{\infty} r^{-2} \\ &< 16(5)^{m(k-m)}\, 2^{-n} V^m. \end{aligned} \tag{74}$$

Now consider the terms with $q = 1$ and $c = 1$. In such a term, we have $N(C) = 1$, and all the elements of C are either 0 or ± 1. So the number of these terms is at most $3^{m(k-m)}$. So, by (71),

$$\sum \left[\!\!\left[\begin{matrix} q=1, \\ c \leq 1, \\ M(C) > 1 \end{matrix}\right]\!\!\right] \left(\frac{N(C)}{q^m}\right)^n I(\nu;\, \mu; C) \leq 3^{m(k-m)}\, 2^{-n} V^m. \tag{75}$$

Combining these inequalities, we see that

$$\sum_{q=1}^{\infty} \sum_{C} \left(\frac{N(C)}{q^m}\right)^n I(\nu; \mu; C) = \sum_{C} \left[\!\!\left[\begin{matrix} q=1, \\ M(C)=1 \end{matrix}\right]\!\!\right] I(\nu;\, \mu;\, C) + R(\nu;\, \mu),$$

where $$0 \leq R(\nu;\, \mu) < 21(5)^{m(k-m)}\, 2^{-n} V^m,$$

* Since a matrix C with $c = l/q$ must have at least one element equal to $\pm l/q$, it follows that the total number of such matrices does not exceed

$$2m(k-m)\,(2l+1)^{m(k-m)-1}.$$

If this bound is used in place of the bound used above the same method leads to the conclusion that the sum (4) converges provided $n \geq m(k-m)+2$, for $m = 1, \ldots, k-1$.

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provided $n > m(k-m)+2$. Now the number of divisions $(\nu;\mu)$ with a fixed value of m is the binomial coefficient

$$\binom{k}{m}.$$

So
$$\sum_{(\nu;\mu)} R(\nu;\mu) < \sum_{m=1}^{k-1} \binom{k}{m} 21(5)^{m(k-m)} 2^{-n} V^m$$
$$< 21(5)^{[\frac{1}{4}k^2]} 2^{-n} (V+1)^k.$$

Thus, provided
$$n > m(k-m)+2,$$

the sum (4) can be written in the form

$$\{\rho(\mathbf{O})\}^k + \left\{\int \rho(\mathbf{X})\,\mathrm{d}\mathbf{X}\right\}^k + \sum_{(\nu;\mu)} \sum_{C} \left[\!\!\left[\begin{matrix} q=1, \\ M(C)=1\end{matrix}\right]\!\!\right] \int \cdots \int \prod_{r=1}^{m} \rho(\mathbf{X}_r) \prod_{s=1}^{k-m} \rho\left(\sum_{i=1}^{m} c_{is}\mathbf{X}_i\right) \mathrm{d}\mathbf{X}_1 \ldots \mathrm{d}\mathbf{X}_m + R_k, \quad (76)$$

where
$$0 \leqslant R_k < 21(5)^{[\frac{1}{4}k^2]} 2^{-n}(V+1)^k. \quad (77)$$

It does not seem to be easy to obtain a good estimate for the integrals $I(\nu;\mu;C)$ in the case when $q=1, M(C)=1$, without making further assumptions about the nature of the function $\rho(\mathbf{X})$. However, we can obtain crude upper and lower bounds for the contributions from these terms, if we assume that $\rho(\mathbf{X})$ is the characteristic function of a set which is symmetrical in the origin. We always have

$$I(\nu;\mu;C) \leqslant V^m.$$

Also, under our assumptions,
$$I(\nu;\mu;C) = V^m$$

if, for $s = 1, \ldots, k-m$, just one of the numbers

$$c_{1s}, c_{2s}, \ldots, c_{ms}$$

has the value ± 1, while the others are zero. The number of matrices C with $q=1, M(C)=1$ is at most

$$3^{m(k-m)},$$

while the number of such matrices of the above type with

$$I(\nu;\mu;C) = V^m$$

is
$$(2m)^{k-m},$$

when $(\nu;\mu) = (1,\ldots,m;\, m+1,\ldots,k)$. So, in the general case,

$$\sum_{C} \left[\!\!\left[\begin{matrix} q=1 \\ M(C)=1\end{matrix}\right]\!\!\right] \left(\frac{N(C)}{q^m}\right)^n I(\nu;\mu;C) \leqslant 3^{m(k-m)} V^m,$$

while for the special division

$$\sum_{C} \left[\!\!\left[\begin{matrix} q=1, \\ M(C)=1\end{matrix}\right]\!\!\right] \left(\frac{N(C)}{q^m}\right)^n I(\nu;\mu;C) \geqslant (2m)^{k-m} V^m.$$

Thus, under these assumptions, the sum (4) can be written in the form

$$J_k + R_k, \quad (78)$$

where

$$J_k = \{\rho(\mathbf{O})\}^k + V^k + \sum_{(\nu;\mu)} \sum_{C} \left[\!\!\left[\begin{matrix} q=1, \\ M(C)=1\end{matrix}\right]\!\!\right] \int \cdots \int \prod_{r=1}^{m} \rho(\mathbf{X}_r) \prod_{s=1}^{k-m} \rho\left(\sum_{i=1}^{m} c_{is}\mathbf{X}_i\right) \mathrm{d}\mathbf{X}_1 \ldots \mathrm{d}\mathbf{X}_m, \quad (79)$$

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and J_k and R_k satisfy

$$\sum_{m=0}^{k}(2m)^{k-m}V^m \leqslant J_k \leqslant \sum_{m=0}^{k}\binom{k}{m}3^{m(k-m)}V^m, \tag{80}$$

$$0 \leqslant R_k < 21(5)^{[\frac{1}{4}k^2]}2^{-n}(V+1)^k. \tag{81}$$

10. In this section S will be taken to be a bounded set, not containing the origin, and having Jordan content V. Take $\rho(\mathbf{X})$ to be the characteristic function of S. Then

$$\rho(\mathbf{O}) = 0, \quad \int \rho(\mathbf{X})\,\mathrm{d}\mathbf{X} = V.$$

So, when $k = 1$, the sum (4) takes the form

$$\rho(\mathbf{O}) + \int \rho(\mathbf{X})\,\mathrm{d}\mathbf{X} = V,$$

and we have

$$\lim_{\omega \to +0}\int_0^1 \ldots \int_0^1 \rho(\Lambda(\alpha_1, \ldots, \alpha_{n-1}, \omega))\,\mathrm{d}\alpha_1 \ldots \mathrm{d}\alpha_{n-1} = V, \tag{82}$$

provided $n \geqslant 2$. The object of this section is to obtain upper and lower bounds for the corresponding limits (3) in the cases when $k = 2$, $n \geqslant 4$ and $k = 3$, $n \geqslant 6$; when $k = 3$, $n \geqslant 6$ we suppose for sake of simplicity that S is symmetrical in $\mathbf{O}$.

When $k = 2$ the integer m must have the value 1 and the only possible divisions are (1; 2) and (2; 1). In each case the matrix C has just one element: in the first case, this may be any rational number, p/q say; in the second case, it must be zero, by condition (6). So the sum (4) takes the form

$$\{\rho(\mathbf{O})\}^2 + \left\{\int \rho(\mathbf{X})\,\mathrm{d}\mathbf{X}\right\}^2 + \sum_{q=1}^{\infty}\sum_{(p,q)=1}\left(\frac{1}{q}\right)^n \int \rho(\mathbf{X})\,\rho\left(\frac{p}{q}\mathbf{X}\right)\mathrm{d}\mathbf{X} + \int \rho(\mathbf{X})\,\rho(\mathbf{O})\,\mathrm{d}\mathbf{X}$$

$$= V^2 + \sum_{q=1}^{\infty}\sum_{\substack{(p,q)=1\\ p\neq 0}}\left(\frac{1}{q}\right)^n \int \rho(\mathbf{X})\,\rho\left(\frac{p}{q}\mathbf{X}\right)\mathrm{d}\mathbf{X}.$$

But, as $\rho(\mathbf{X})$ is the characteristic function of a set, we have

$$\left.\begin{aligned} &0 \leqslant \rho(\mathbf{X})\,\rho(\pm\mathbf{X}) \leqslant \rho(\mathbf{X}), \\ &0 \leqslant \rho(\mathbf{X})\,\rho\left(\frac{p}{q}\mathbf{X}\right) \leqslant \rho(\mathbf{X}) \quad \text{if} \quad |p| < q, \\ &0 \leqslant \rho(\mathbf{X})\,\rho\left(\frac{p}{q}\mathbf{X}\right) \leqslant \rho\left(\frac{p}{q}\mathbf{X}\right) \quad \text{if} \quad |p| > q. \end{aligned}\right\} \tag{83}$$

Thus, the sum (4) can be written in the form

$$J_2 + R_2,$$

where

$$J_2 = V^2 + \int \rho(\mathbf{X})\,\mathrm{d}\mathbf{X} + \int \rho(\mathbf{X})\,\rho(-\mathbf{X})\,\mathrm{d}\mathbf{X} = V^2 + V + \int \rho(\mathbf{X})\,\rho(-\mathbf{X})\,\mathrm{d}\mathbf{X},$$

and

$$0 \leqslant R_2 \leqslant \sum_{q=2}^{\infty}\sum_{\substack{(p,q)=1\\ |p|<q}}\left(\frac{1}{q}\right)^n \int \rho(\mathbf{X})\,\mathrm{d}\mathbf{X} + \sum_{q=1}^{\infty}\sum_{\substack{(p,q)=1\\ |p|>q}}\left(\frac{1}{q}\right)^n \int \rho\left(\frac{p}{q}\mathbf{X}\right)\mathrm{d}\mathbf{X}.$$

Here the upper bound for R_2 can be rewritten in the form

$$2V\sum_{q=2}^{\infty}\sum_{\substack{1\leqslant p<q\\(p,q)=1}} q^{-n}+2V\sum_{q=1}^{\infty}\sum_{\substack{p=q+1\\(p,q)=1}}^{\infty}\left(\frac{1}{q}\right)^n\left(\frac{q}{p}\right)^n = 2V\sum_{q=2}^{\infty}\sum_{\substack{1\leqslant p<q\\(p,q)=1}} q^{-n}+2V\sum_{p=2}^{\infty}\sum_{\substack{1\leqslant q<p\\(q,p)=1}} p^{-n}$$

$$= 4V\sum_{q=2}^{\infty}\phi(q)\,q^{-n}$$

$$= 4V\left[\frac{\zeta(n-1)}{\zeta(n)}-1\right],$$

provided $n\geqslant 3$. Thus, if $n\geqslant 3$,

$$0\leqslant R_2\leqslant 4V\left[\frac{\zeta(n-1)}{\zeta(n)}-1\right].$$

It is worth noticing that, if S is the set of points other than the origin of a bounded symmetrical star set, then the results (83) hold with equality in the right-hand inequalities, so that, in this case,

$$J_2+R_2 = V^2+4V\left[\frac{\zeta(n-1)}{\zeta(n)}-\frac{1}{2}\right].$$

It follows from theorem 1 that

$$\lim_{\omega\to+0}\int_0^1\cdots\int_0^1\{\rho(\Lambda(\alpha_1,\ldots,\alpha_{n-1},\omega))\}^2\,d\alpha_1\ldots d\alpha_{n-1} = V^2+V+\int\rho(\mathbf{X})\,\rho(-\mathbf{X})\,d\mathbf{X}+R_2, \tag{84}$$

where

$$0\leqslant R_2\leqslant 4V\left[\frac{\zeta(n-1)}{\zeta(n)}-1\right], \tag{85}$$

provided $n\geqslant 3$, and the equality holds in the right-hand inequality of (85) when S is the set of points other than the origin of a bounded symmetrical star set.

When $k=3$, the number m may have the value 1 or 2. In each case C will be a matrix with just two elements. We first consider the matrices C with $q=1$ and $M(C)=1$. The possible divisions and matrices (written as row matrices) are

$(1;\,2,3)$: $(0,0), \pm(1,0), \pm(0,1), \pm(1,1), \pm(1,-1)$;
$(2;\,1,3)$: $(0,0), \pm(0,1)$;
$(3;\,1,2)$: $(0,0)$;
$(1,2;\,3)$: $(0,0), \pm(1,0), \pm(0,1), \pm(1,1), \pm(1,-1)$;
$(1,3;\,2)$: $(0,0), \pm(1,0)$;
$(2,3;\,1)$: $(0,0)$.

So, in this case, assuming that S is symmetrical in $\mathbf{O}$, the sum (79) becomes

$$\begin{aligned}J_3 &= V^3+\sum_{\pm\pm}\int\rho(\mathbf{X})\,\rho(\pm\mathbf{X})\,\rho(\pm\mathbf{X})\,d\mathbf{X}+3\sum_{\pm}\iint\rho(\mathbf{X}_1)\,\rho(\mathbf{X}_2)\,\rho(\pm\mathbf{X}_1)\,d\mathbf{X}_1\,d\mathbf{X}_2\\ &\qquad+\sum_{\pm\pm}\iint\rho(\mathbf{X}_1)\,\rho(\mathbf{X}_2)\,\rho(\pm\mathbf{X}_1\pm\mathbf{X}_2)\,d\mathbf{X}_1\,d\mathbf{X}_2\\ &= V^3+6V^2+4\iint\rho(\mathbf{X}_1)\,\rho(\mathbf{X}_2)\,\rho(\mathbf{X}_1+\mathbf{X}_2)\,d\mathbf{X}_1\,d\mathbf{X}_2+4V.\end{aligned}$$

Thus, provided $n\geqslant 6$, the limit

$$\lim_{\omega\to+0}\int_0^1\cdots\int_0^1\{\rho(\Lambda(\alpha_1,\ldots,\alpha_{n-1},\omega))\}^3\,d\alpha_1\ldots d\alpha_{n-1}$$

exists and has the value

$$V^3+6V^2+4\iint\rho(\mathbf{X}_1)\,\rho(\mathbf{X}_2)\rho(\mathbf{X}_1+\mathbf{X}_2)\,\mathrm{d}\mathbf{X}_1\,\mathrm{d}\mathbf{X}_2+4V+R_3, \tag{86}$$

where, by (81),

$$0\leqslant R_3<525(V+1)^3\,(\tfrac{1}{2})^n. \tag{87}$$

Here one has trivially

$$0\leqslant\iint\rho(\mathbf{X}_1)\,\rho(\mathbf{X}_2)\,\rho(\mathbf{X}_1+\mathbf{X}_2)\,\mathrm{d}\mathbf{X}_1\,\mathrm{d}\mathbf{X}_2\leqslant V^2. \tag{88}$$

11. This section uses the results of the last section to give the following

Proof of theorem 4. We suppose that there is no lattice with determinant 1, which has no point in S. Then by the symmetry of S, every lattice with determinant 1 has at least two points in S. So, if $\rho(\mathbf{X})$ is the characteristic function of the set S, we have

$$\rho(\Lambda(\alpha_1,\ldots,\alpha_{n-1},\omega))\geqslant 2$$

for all $\alpha_1,\ldots,\alpha_{n-1}$ and $\omega>0$. Thus

$$\lim_{\omega\to 0}\int_0^1\ldots\int_0^1\{\rho(\Lambda(\alpha_1,\ldots,\alpha_{n-1},\omega))-2\}\,\mathrm{d}\alpha_1\ldots\mathrm{d}\alpha_{n-1}\geqslant 0.$$

Hence, by (82),

$$V-2\geqslant 0,$$

and $V\geqslant 2$.

But, writing

$$\rho=\rho(\Lambda(\alpha_1,\ldots,\alpha_{n-1},\omega)),$$

for simplicity, it is clear that

$$\lim_{\omega\to+0}\int_0^1\ldots\int_0^1\{\rho-2\}\{\lambda+\mu\rho\}^2\,\mathrm{d}\alpha_1\ldots\mathrm{d}\alpha_{n-1}\geqslant 0.$$

So, writing,

$$\mu_k=\lim_{\omega\to+0}\int_0^1\ldots\int_0^1\rho^k\,\mathrm{d}\alpha_1\ldots\mathrm{d}\alpha_{n-1},$$

we have

$$(\mu_1-2\mu_0)\,\lambda^2+2(\mu_2-2\mu_1)\,\lambda\mu+(\mu_3-2\mu_2)\,\mu^2\geqslant 0,$$

for all real λ, μ. Consequently

$$(\mu_1-2\mu_0)\,(\mu_3-2\mu_2)-(\mu_2-2\mu_1)^2\geqslant 0.$$

Now, by (82), (84), (85), (86), (88) and (87),

$$\begin{aligned}\mu_0&=1,\\ \mu_1&=V,\\ \mu_2&=V^2+2V+R_2,\\ \mu_3&\leqslant V^3+10V^2+4V+R_3,\end{aligned}$$

where

$$\left.\begin{aligned}&0\leqslant R_2\leqslant 2V\left[\frac{2\zeta(n-1)}{\zeta(n)}-2\right],\\ &0\leqslant R_3<525(V+1)^3\,(\tfrac{1}{2})^n.\end{aligned}\right\} \tag{89}$$

Thus

$$\begin{aligned}\mu_1-2\mu_0&=V-2,\\ \mu_2-2\mu_1&=V^2+R_2,\\ \mu_3-2\mu_2&\leqslant V^3+8V^2+R_3,\end{aligned}$$

as $R_2\geqslant 0$. Consequently, since $V\geqslant 2$,

$$(V-2)\,(V^3+8V^2+R_3)-(V^2+R_2)^2\geqslant 0.$$

This implies that

$$(V-2)(V^3+8V^2+R_3)-(V-2)(V^3+2V^2)-4V^2\geqslant 0,$$

so that

$$V\geqslant 2+4V^2[6V^2+R_3]^{-1}$$

$$=2+\frac{2}{3}\left[1+\frac{R_3}{6V^2}\right]^{-1}.$$

But we have assumed that $V<2+\frac{2}{3}$ and proved that $V\geqslant 2$. Hence, by (89),

$$\frac{R_3}{6V^2}<\frac{525}{6}\frac{(V+1)^3}{V^2}(\tfrac{1}{2})^n<\frac{525}{6}\frac{(3\frac{2}{3})^3}{(2\frac{2}{3})^2}(\tfrac{1}{2})^n<633(\tfrac{1}{2})^n,$$

since

$$\frac{d}{dV}\frac{(V+1)^3}{V^2}\geqslant 0 \quad \text{if} \quad V\geqslant 2.$$

Thus

$$V\geqslant 2+\tfrac{2}{3}[1+633(\tfrac{1}{2})^n]^{-1},$$

contrary to our hypotheses. This contradiction proves the theorem.

References

Hlawka, E. 1944 *Math. Z.* **49**, 285.
Mahler, K. 1949 *Proc. Acad. Sci. Amst.* **52**, 633.
Siegel, C. L. 1945 *Ann. Math., Princeton*, **46**, 340.